

**REPORT NO.
UCD/CGM-26/01**

CENTER FOR GEOTECHNICAL MODELING

**PM4SAND3D (VERSION 1.0):
A SAND PLASTICITY MODEL FOR
EARTHQUAKE ENGINEERING
APPLICATIONS**

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June 2026

**PM4Sand3D (Version 1):
A Sand Plasticity Model for Earthquake Engineering Applications**

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Report No. UCD/CGM-26-01

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June 2026

First Edition (PM4Sand3D Version 1.0) June 2026

ABSTRACT

The sand plasticity model PM4Sand3D for geotechnical earthquake engineering applications is presented. PM4Sand3D is a three-dimensional (3D) version of the two-dimensional (2D) model PM4Sand v3.3. The model follows the basic framework of the stress ratio-controlled, critical state-compatible, bounding surface plasticity model for sand presented by Dafalias and Manzari (2004). Modifications to the model were developed and implemented by Boulanger (2010, version 1), Boulanger and Ziotopoulou (2012, version 2), Boulanger and Ziotopoulou (2015, version 3), and Boulanger and Ziotopoulou (2023, version 3.3) to improve its ability to approximate the stress-strain responses important to geotechnical earthquake engineering applications; in essence, the model was calibrated at the equation level to provide for better approximation of the trends observed across a set of experimentally- and case history-based design correlations. These constitutive modifications included: revising the fabric formation/destruction to depend on plastic shear rather than plastic volumetric strains; adding fabric history and cumulative fabric formation terms; modifying the plastic modulus relationship and making it dependent on fabric; modifying the dilatancy relationships to include dependence on fabric and fabric history, and to provide more distinct control of volumetric contraction versus expansion behavior; providing a constraint on the dilatancy during volumetric expansion so that it is consistent with Bolton's (1986) dilatancy relationship; modifying the elastic modulus relationship to include dependence on stress ratio and fabric history; modifying the logic for tracking previous initial back-stress ratios (i.e., loading history effect); recasting the critical state framework in terms of a relative state parameter index; incorporating a methodology for improved modeling of post-liquefaction reconsolidation strains; and providing default values for all but three primary input parameters. These modifications are contained within PM4Sand3D, presented herein, with revisions to the numerical implementation and the addition of a Lode angle dependency. The model is coded as a user-defined material in a dynamic link library (DLL) for use with the commercial program FLAC3D 9.X (Itasca 2024). The numerical implementation and DLL module are described. The behavior of the model is illustrated by simulations of element loading tests covering a range of conditions, including drained and undrained, cyclic and monotonic loading under a range of initial confining and shear stress conditions, which can then be compared to typical design relationships. The model is shown to provide reasonable approximations of desired behaviors and to be relatively easy to calibrate.

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PM4Sand3D (Version 1): A Sand Plasticity Model for Earthquake Engineering Applications

1. INTRODUCTION

Nonlinear dynamic analyses (NDAs) for problems involving liquefaction are common for larger projects in earthquake engineering practice. Constitutive models for sand that have been used in practice range from relatively simplified, uncoupled cycle-counting models to more complex plasticity models (e.g., Wang et al. 1990, Cubrinovski and Ishihara 1998, Dawson et al. 2001, Papadimitriou et al. 2001, Yang et al. 2003, Byrne et al. 2004, Dafalias and Manzari 2004, Cheng et al. 2013, Yang et al. 2020, Cheng and Detournay 2021). Each constitutive model has certain advantages and limitations that can be illustrated for potential users by documents showing the constitutive response of the model in element tests that cover a broad range of the conditions that may be important to various applications in practice (e.g., Boulanger and Beaty 2016, Boulanger 2022).

The information available for calibration of constitutive models in design practice most commonly consists of basic classification index tests (e.g., grain size distributions), in-situ penetration resistances (e.g., SPT or CPT), and shear wave velocity (V_s) measurements. More advanced laboratory testing, such as triaxial or direct simple shear tests, is almost never available due to problems related to mitigating sampling disturbance effects and the challenge of identifying representative samples from highly heterogeneous deposits.

Constitutive models for geotechnical earthquake engineering applications must approximate a broad range of field conditions. For example, a single geotechnical structure like the schematic earth dam shown in Figure 1.1, may include strata or zones of sand ranging from very loose to dense under a wide range of confining stresses, initial static shear stresses (e.g., at different points beneath the slope), drainage conditions (e.g., above and below the water table), and seismic loading intensities. Engineering effort is significantly reduced when the constitutive model can reasonably stress-strain behaviors across these ranges. If the model cannot approximate the trends across all these conditions, additional engineering effort is required to decide what behaviors should be prioritized in the calibration process or to develop additional/alternative calibrations.

In addition, some geotechnical applications may warrant the effort of modeling in three dimensions (3D) to approximate their principal features. Figure 1.2 illustrates situations in which behaviors may not be represented through plane-strain simplifications. These include complex soil conditions, embankments in narrow canyons, pile foundation systems, triaxial tests, and multidimensional loading caused by earthquake ground motions.

The PM4Sand3D (version 1) plasticity model for geotechnical earthquake engineering applications is presented herein. This model is a 3D extension of the plane-strain model PM4Sand v3.3 (Boulanger and Ziotopoulou 2023). The PM4Sand model follows the basic framework of the stress ratio-controlled, critical state-compatible, bounding surface plasticity model for sand initially presented by Manzari and Dafalias (1997) and later extended by Dafalias and Manzari (2004). Modifications to the Dafalias-

Manzari model were developed and implemented to improve its ability to approximate engineering design relationships used to estimate stress-strain behaviors that are important for predicting liquefaction-induced ground deformations during earthquakes. These developments of PM4Sand are described in the manuals for each version of the model (Boulanger 2010, Boulanger and Ziotopoulou 2012, 2015, 2017, 2022, and 2023) and associated publications (Boulanger and Ziotopoulou 2013, Ziotopoulou and Boulanger 2013a, Ziotopoulou and Boulanger 2016). The extension of the model to 3D conditions is described herein. The PM4Sand3D model is coded as a dynamic link library (DLL) for use with the commercial program FLAC3D 9.X (Itasca 2024).

Development of PM4Sand3D, relative to PM4Sand version 3.3, involved the following tasks. Lode angle dependency was incorporated following the derivation in Dafalias and Manzari (2004) in combination with the formulation from Loukidis and Salgado (2009), along with rederivation of the default relationship for the dilatancy parameters in three dimensions to maintain consistency with stress-dilatancy laws. Other changes were made for numerical implementation in 3D rather than 2D, including generalization of the tracking logic for back-stress ratios and reversals, the rigid body rotation algorithm, and the overloaded functions in the tensor library. The functional forms of the equations, such as for dilatancy and plastic modulus, remain the same as for PM4Sand. Lastly, a new generalized calibration was performed to determine default values for secondary parameters in the revised model.

It is unlikely that any one model can be developed or calibrated to simultaneously fit a full set of applicable design correlations for monotonic and cyclic, drained and undrained behaviors of sand, in part because the various design correlations are not necessarily physically consistent with each other; e.g., they may include a mix of laboratory test-based and case history-based relationships, or they have been empirically derived from laboratory data sets for different sands. Nonetheless, it is desirable that a model, after calibration to the design relationship that is of primary importance to a specific project, be able to produce behaviors that are reasonably consistent with the general magnitudes and trends in other applicable design correlations or typical experimental data.

Stress-strain behaviors of sand that are most commonly the focus in design are listed below, along with reference to a figure showing an example design correlation or typical experimental test result.

- The cyclic resistance ratio (CRR) against triggering of liquefaction, which is commonly estimated based on SPT and CPT penetration resistances with case-history-based liquefaction correlations (e.g., Figure 1.3). The CRR is the cyclic stress ratio (e.g., $CSR = \tau_{cyc}/\sigma'_{vc}$, with τ_{cyc} = horizontal cyclic shear stress, σ'_{vc} = vertical consolidation stress) that is required to trigger liquefaction in a specified number of equivalent uniform loading cycles.
- The response under the irregular cyclic loading histories produced by earthquakes, which is approximately represented by the relationship between CRR and number of equivalent uniform loading cycles (e.g., Figure 1.4). This aspect of behavior also directly relates to the magnitude scaling factors (MSF) that are used with liquefaction correlations in practice.
- The dependence of CRR on effective confining stresses and sustained static shear stresses. These aspects of behavior are represented by the K_σ (Figure 1.5) and K_α (Figure 1.6) relationships, respectively, which are used with liquefaction correlations in practice.
- The accumulation of shear strains after triggering liquefaction. Evaluations of reasonable behavior are often based on comparisons to laboratory tests results for similar soils in the literature (e.g., Figure 1.7).

- The small-strain shear modulus which can be obtained through in-situ shear wave velocity measurements.
- The shear modulus reduction and equivalent damping ratio relationships prior to triggering of liquefaction. These aspects of behavior are commonly estimated using empirical correlations derived from laboratory test results for similar soils in the literature (e.g., Figure 1.8).
- Drained monotonic shear strengths and stress-strain behavior (e.g., Figure 1.9). Peak friction angles may be estimated using relationships such as Bolton's (1986) relative dilatancy index, I_R (Figure 1.10) or correlations to SPT and CPT penetration resistances.
- Undrained monotonic shear strengths and stress-strain behavior (e.g., Figure 1.11), which may be estimated using correlations to SPT and CPT penetration resistances.
- The strength loss due to liquefaction, which may involve explicitly modeling phenomena such as void redistribution or empirically accounting for it through case history-based residual strength correlations (e.g., Figure 1.12).
- The volumetric strains during drained cyclic loading (Figure 1.13 and Figure 1.14) or due to reconsolidation following triggering of liquefaction (e.g., Figure 1.15), both of which may be estimated using empirical correlations derived from laboratory test results for similar soils in the literature.

Another important consideration in the evaluation of 3D soil models is their ability to approximate the stress-strain responses of sand under more general 3D loading paths. Two loading responses examined herein are listed below, with reference to illustrative figures.

- For monotonic drained loading, the dependence of the peak friction angle on the intermediate principal stress (Figures 1.16 and 1.17). This aspect of behavior is generally estimated using empirical trends derived from laboratory test results, such as shown in these figures.
- For cyclic undrained loading, the effect of multidirectional loading on triggering resistance and shear strain accumulation. Bidirectional direct simple shear tests have examined responses with and without static shear biases; tests with the cyclic loading perpendicular to a static shear bias (Figure 1.18) have produced slightly lower CRRs than tests with the cyclic loading parallel to the static shear bias (Figure 1.19). Other studies have provided data showing responses to rotational and other bidirectional loading patterns (e.g., Kammerer et al. 2002, Bhaumik 2018).

The organization of this report is as follows:

- Section 2 of this report contains a description of the model formulation.
- Section 3 contains a description of the model's implementation as a user-defined material in a dynamic link library for use in the commercial program FLAC3D 9.X (Itasca 2024).
- Section 4 of this report contains a summary of the model input parameters, guidance on model parameter selections, and illustrations of the model responses to a range of elemental loading conditions.
- Section 5 contains summary remarks regarding the model and its use in practice.

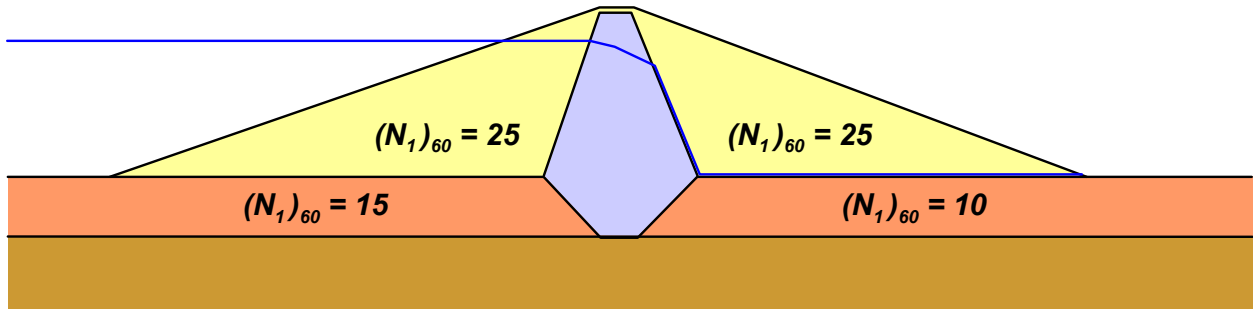


Figure 1.1. Schematic cross-section for an earth dam.

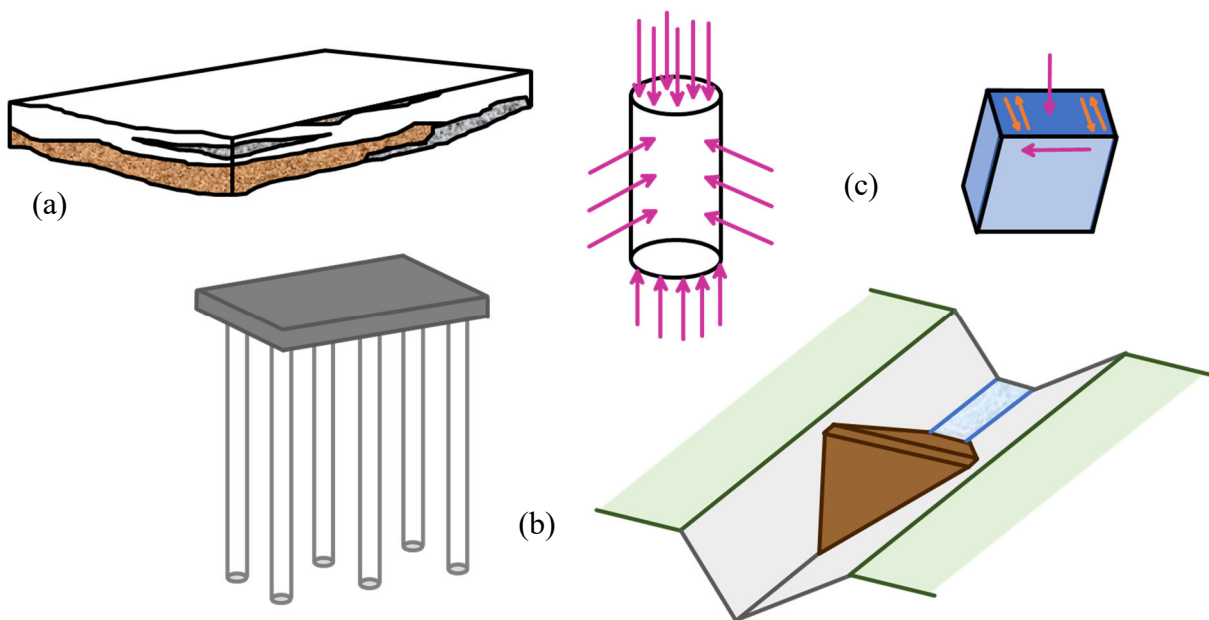


Figure 1.2. Schematics of conditions requiring 3D modeling, due to (a) complex soil conditions, (b) structure geometry, or (c) multidirectional loading.

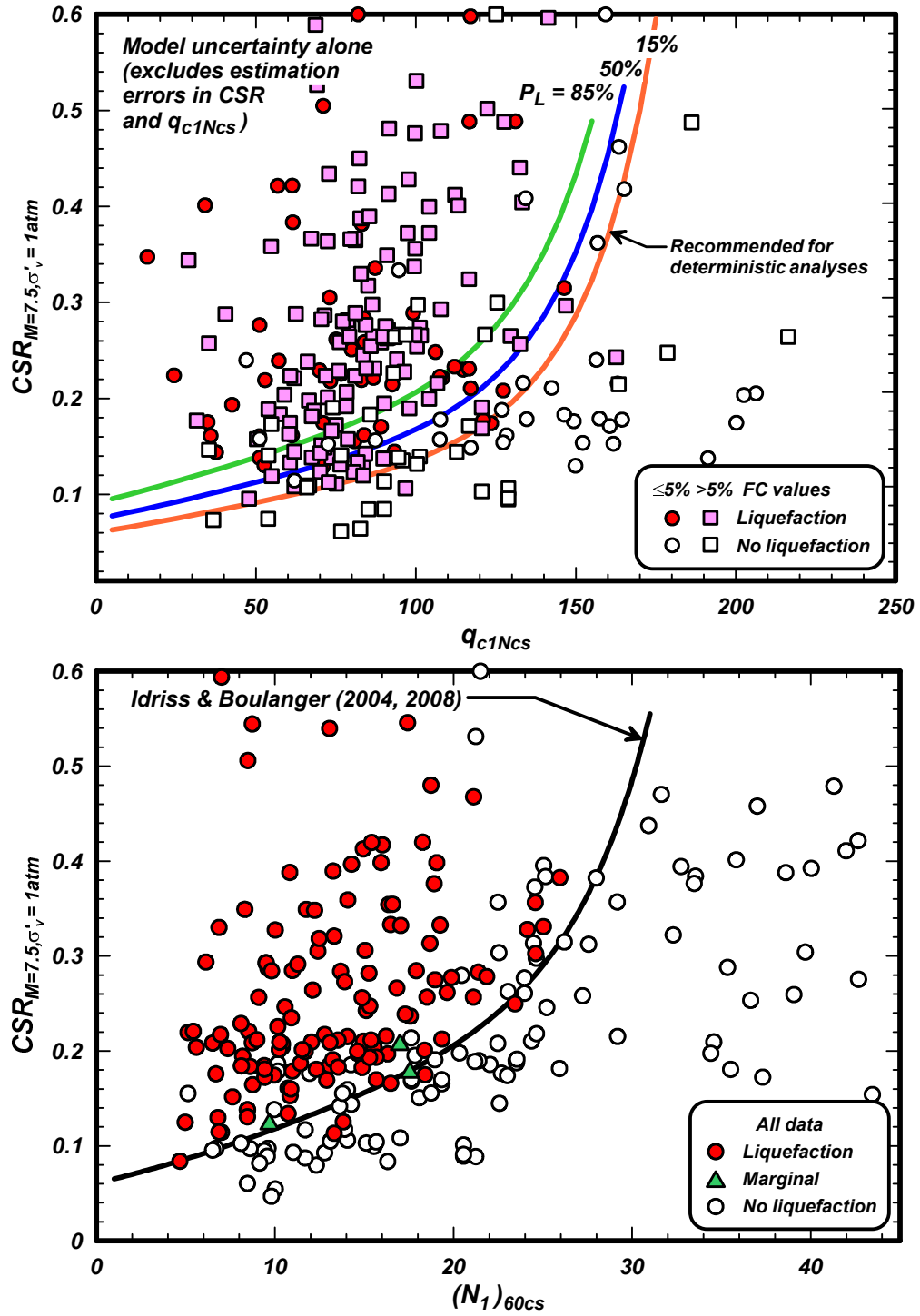


Figure 1.3. Correlations for cyclic resistance ratio (CRR) of sand-like soils from CPT and SPT data using the procedures by Boulanger and Idriss (2014).

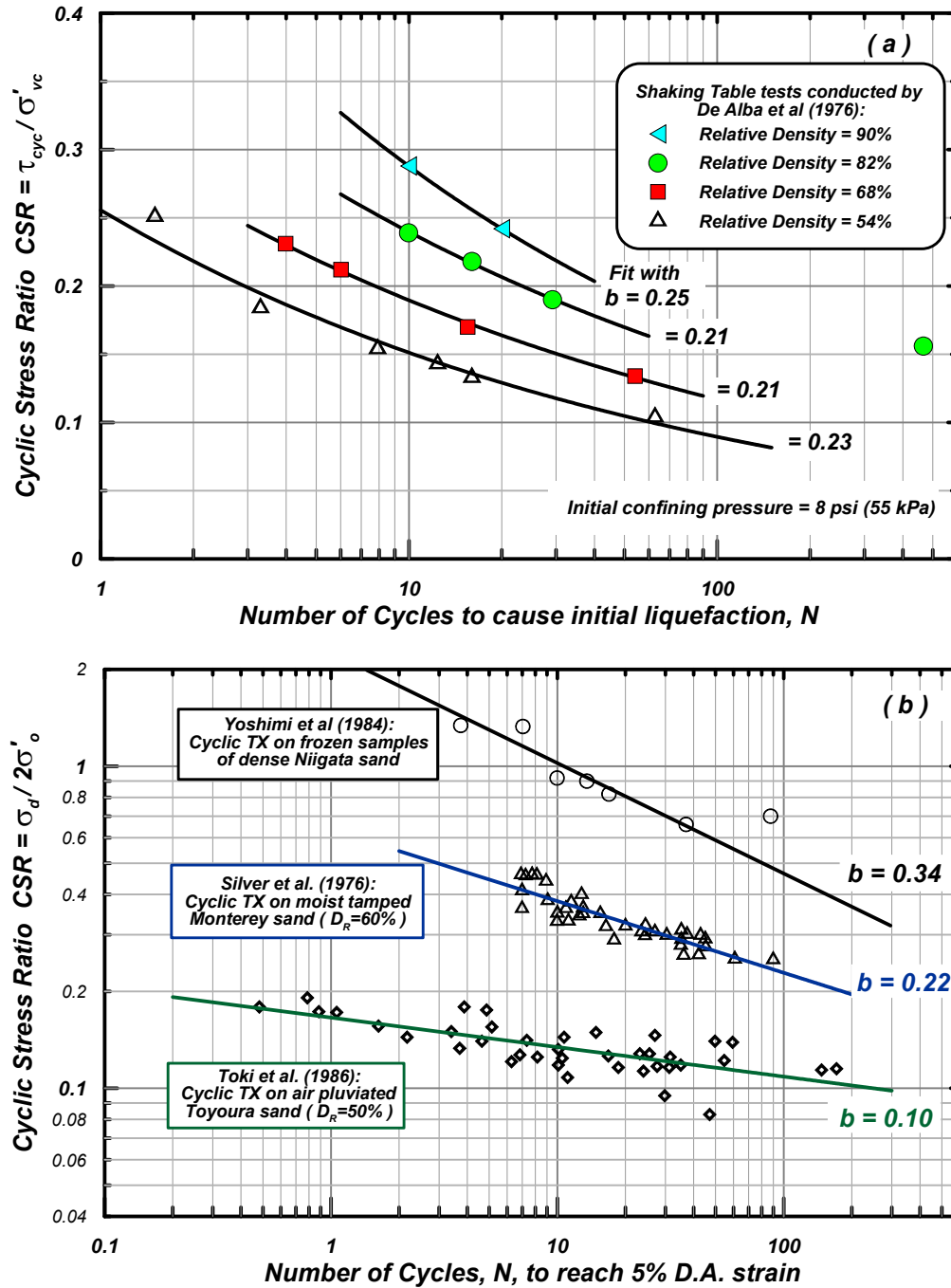


Figure 1.4. Relationships between cyclic resistance ratio (CRR) and number of equivalent uniform loading cycles for undrained loading of reconstituted and undisturbed samples of clean sand (from Ziotopoulou and Boulanger 2012).

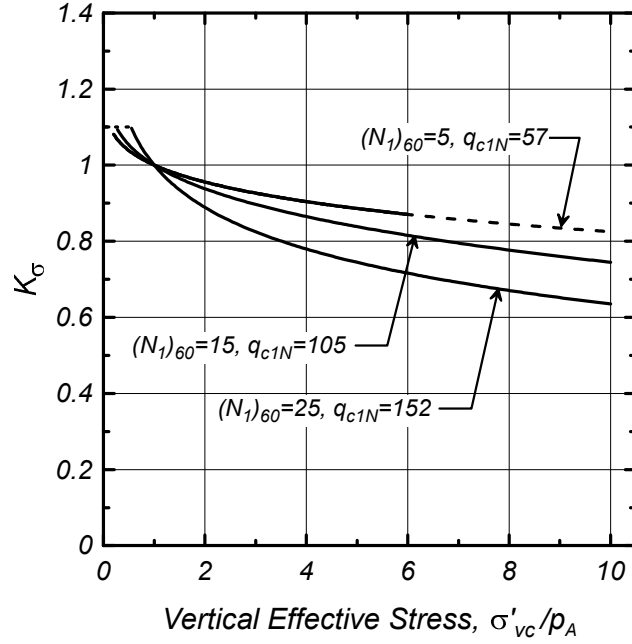


Figure 1.5. K_σ factor describing the effect that effective overburden stress has on cyclic resistance ratio of sands (from Idriss and Boulanger 2008).

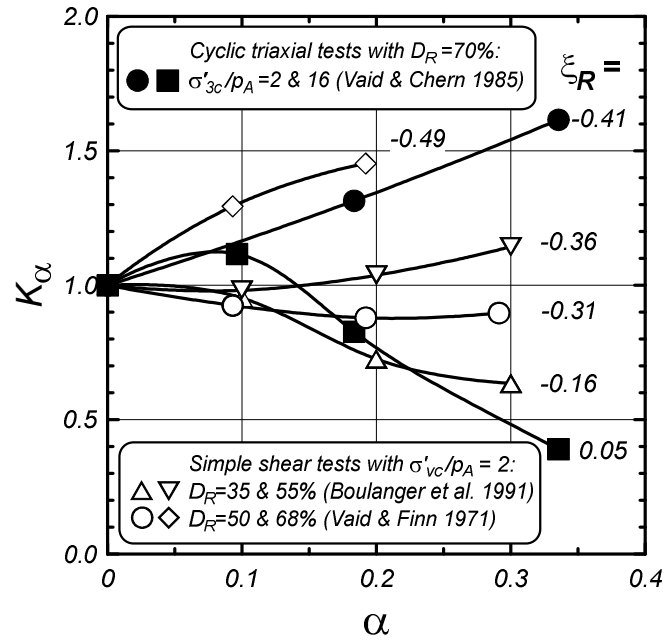


Figure 1.6. K_α factor describing the effect that sustained static shear stress ratio ($\alpha=\tau_s/\sigma'_{vc}$) has on cyclic resistance ratio of sands (Boulanger 2003a).

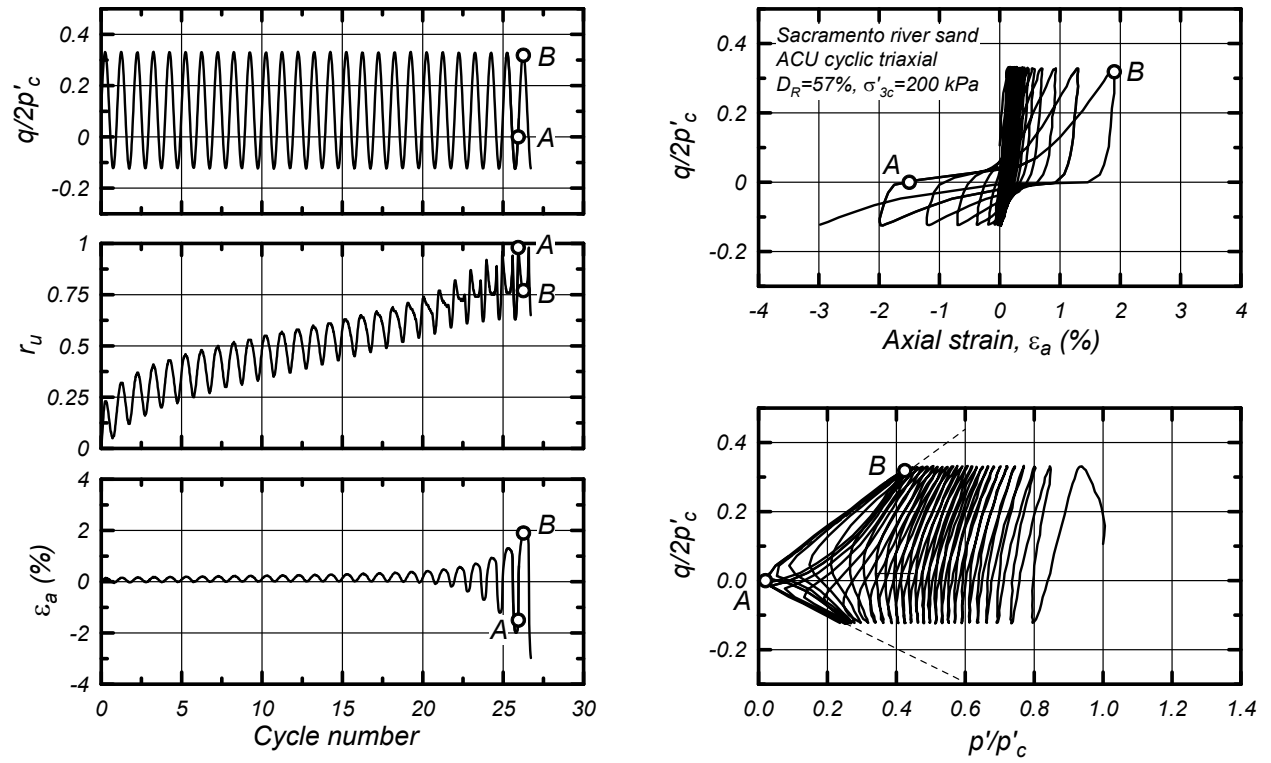


Figure 1.7. Undrained cyclic triaxial test on clean sand (test from Boulanger and Truman 1996; from Idriss and Boulanger 2008).

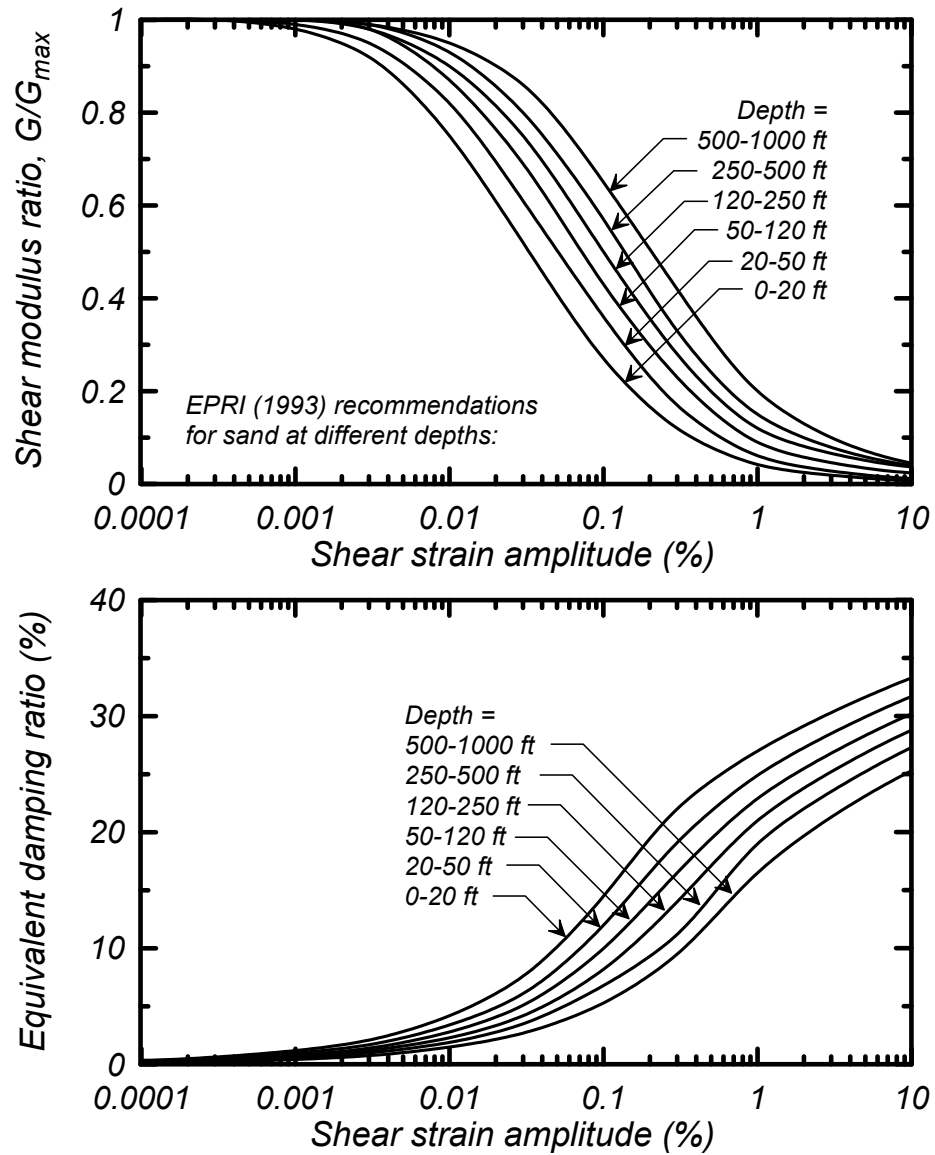


Figure 1.8. Shear modulus reduction and equivalent damping ratio relationship for sands, as recommended by EPRI (1993).

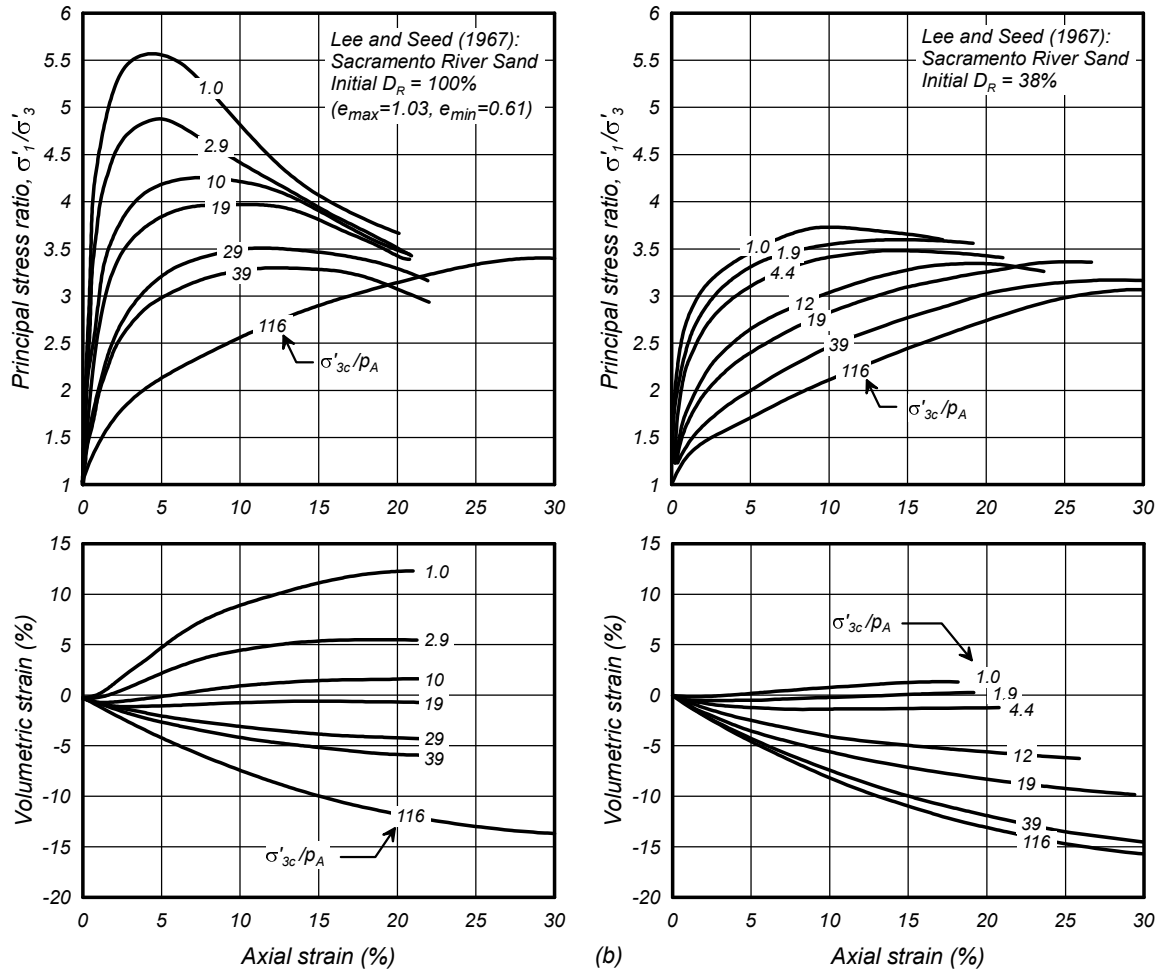


Figure 1.9. Drained triaxial compression tests on loose and dense sand specimens under a range of effective confining stresses (after Lee and Seed 1967; from Idriss and Boulanger 2008).

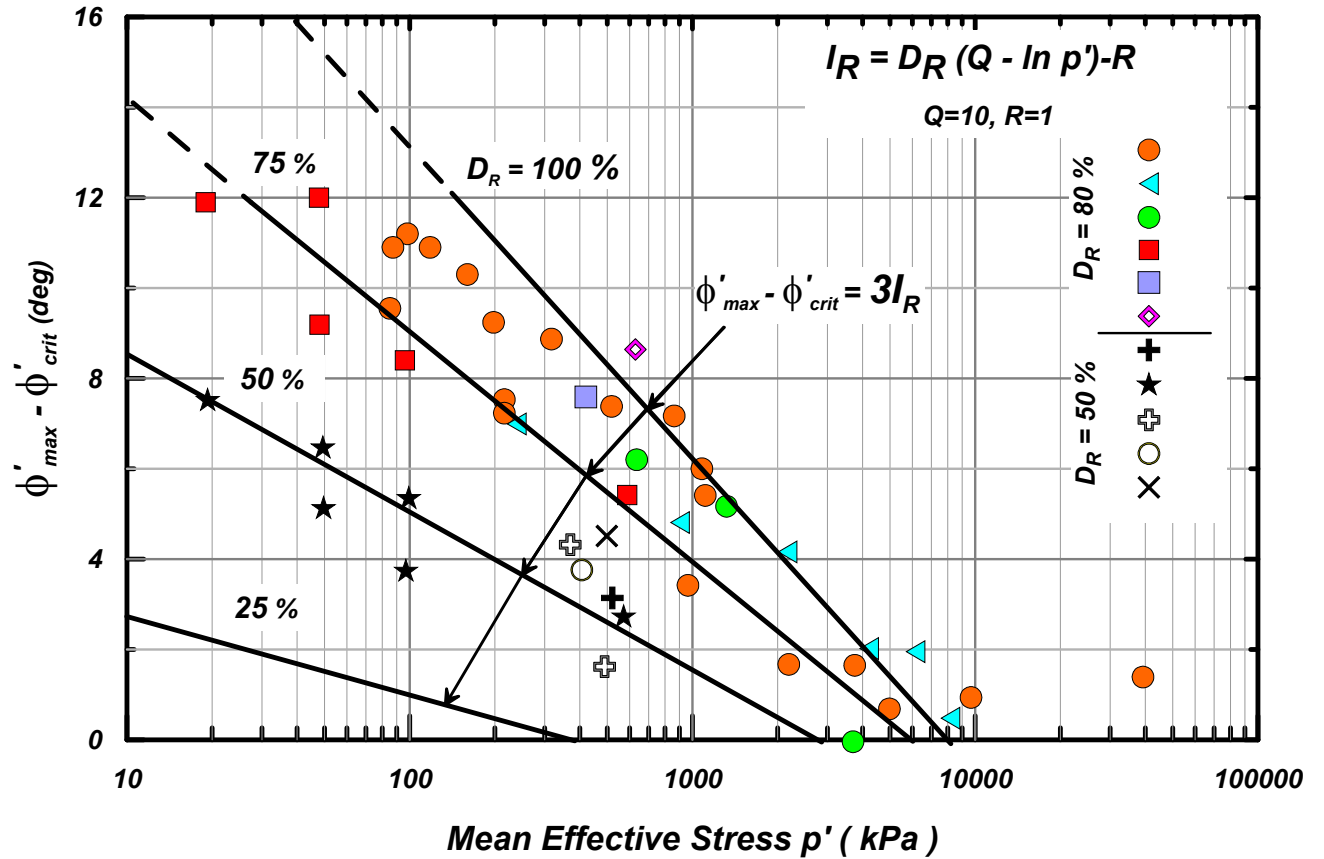


Figure 1.10. Triaxial test data for sands with initial relative densities in the vicinity of 80% or 50% failing at various mean effective stresses. The difference of peak friction angle from the critical friction angle is related to the relative dilatancy index (I_R) (after Bolton 1986).

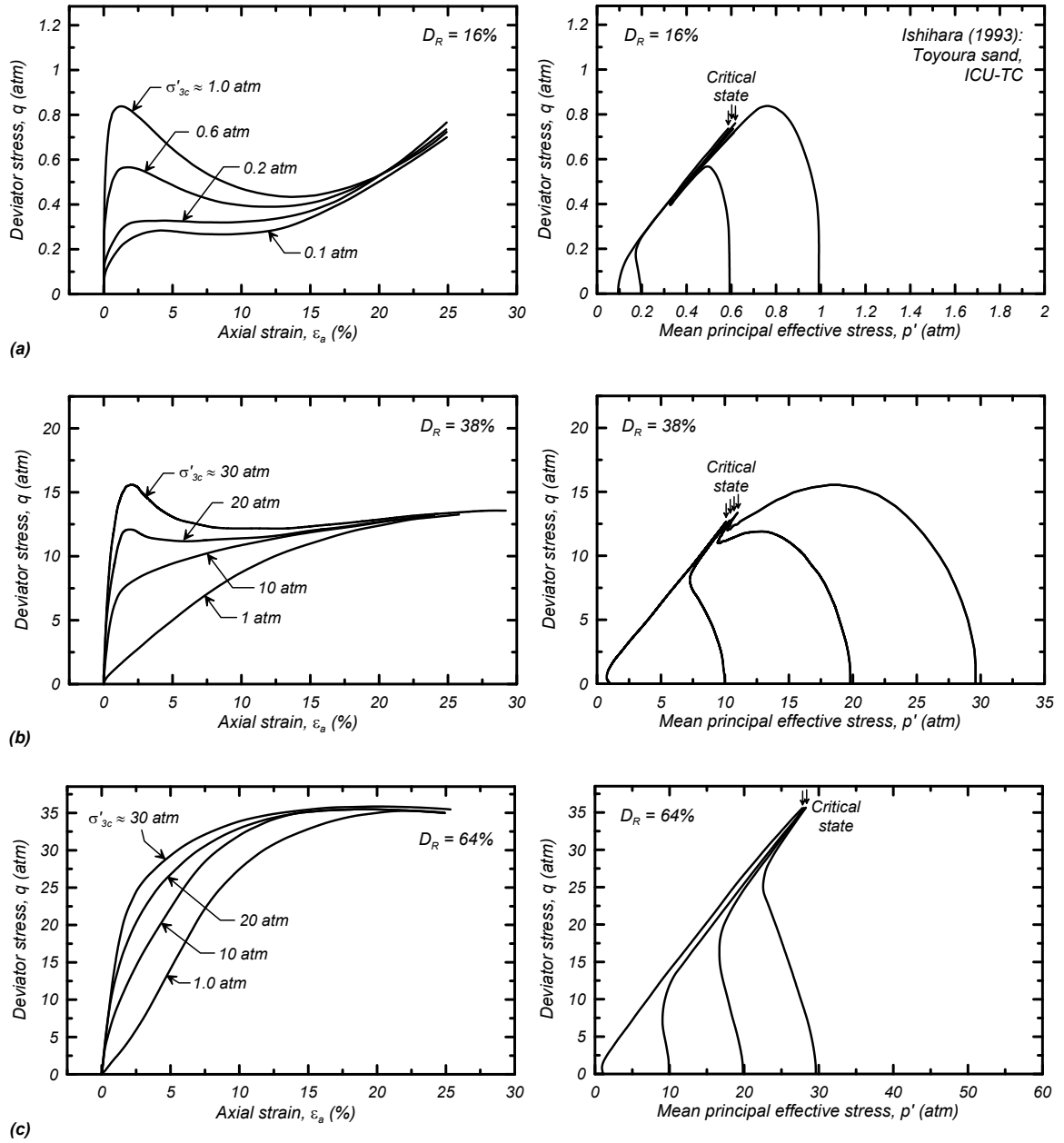


Figure 1.11. Undrained triaxial compression tests on very loose and loose sand specimens under a range of effective consolidation stresses (after Ishihara 1993; from Idriss and Boulanger 2008).

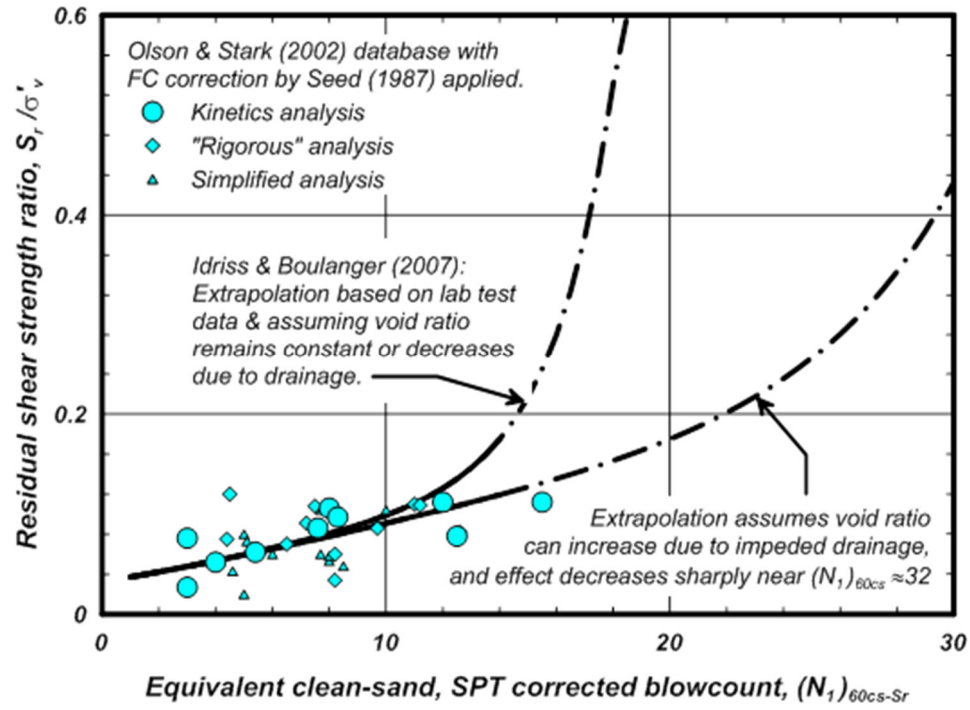


Figure 1.12. Empirical relationship for estimating residual strength of liquefied sands based on case histories (Boulanger and Idriss 2011).

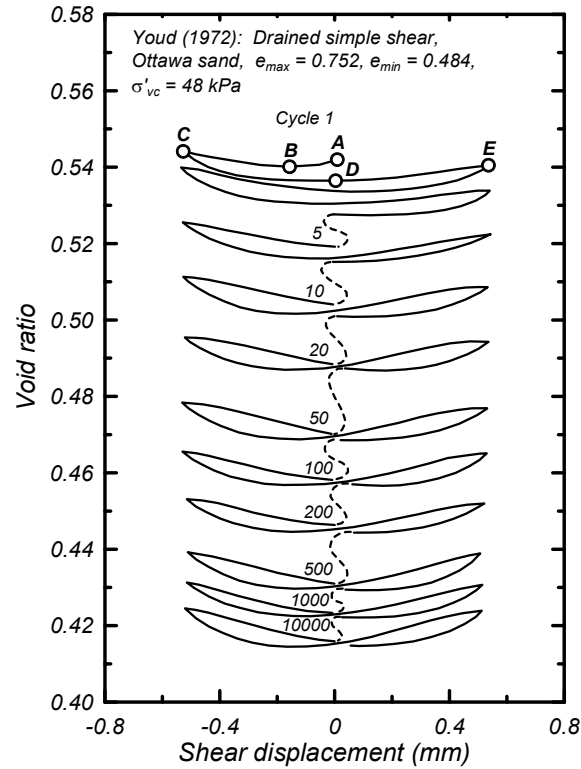


Figure 1.13. Drained cyclic simple shear test showing densification of a sand specimen with successive cycles of loading (after Youd 1972; from Idriss and Boulanger 2008).

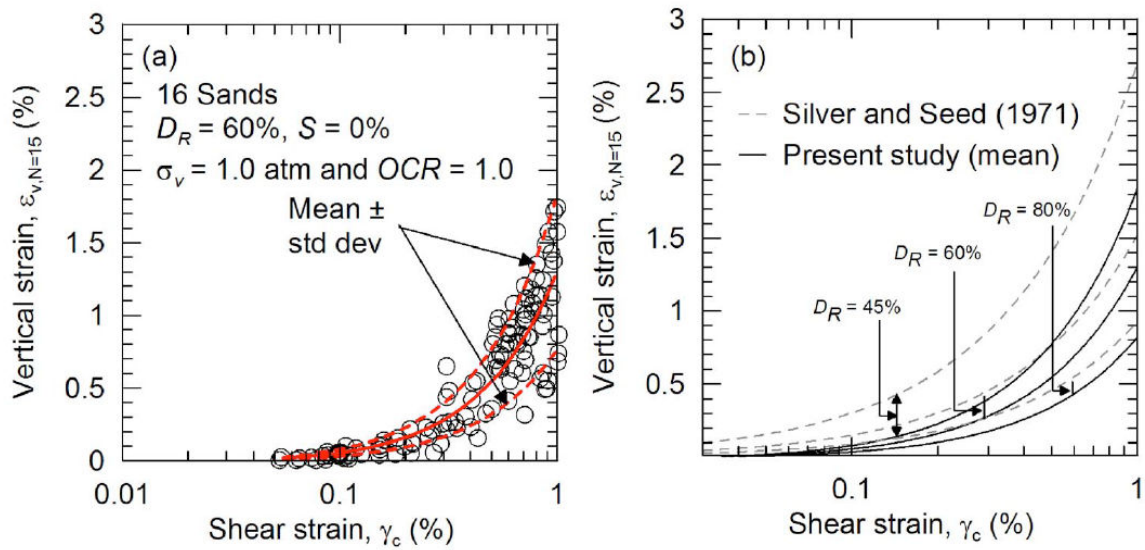


Figure 1.14. Volumetric strains in drained cyclic direct simple shear tests on clean sands (Duku et al. 2008): (a) Results from 16 sands at a relative density of about 60% with an overburden stress of 1.0 atm, and (b) Comparison of trends with earlier relationships by Silver and Seed (1971) for sands at relative densities of 45, 60, and 80%.

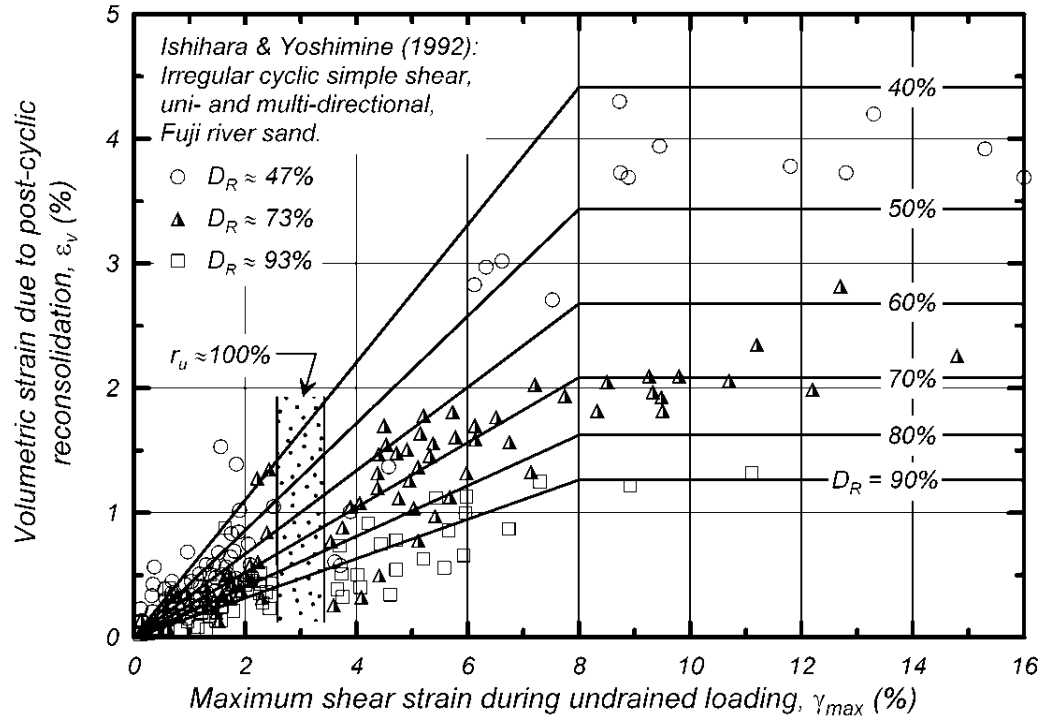
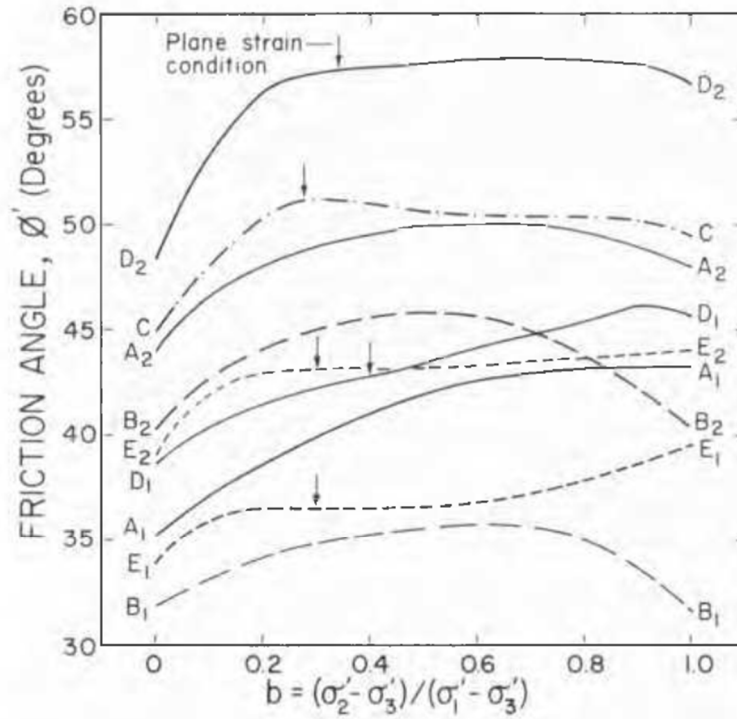


Figure 1.15. Relationship between post-liquefaction volumetric strain and the maximum shear strain induced during undrained cyclic loading of clean sand (after Ishihara and Yoshimine 1992; redrawn in Idriss and Boulanger 2008).



LINE	SAND	D_r (%)	REFERENCE
A ₁	Ottawa	med. loose	Ko and Scott (1968)
A ₂		med. dense	
B ₁	Med Fine Loch Aline	30	Sutherland and Mesdary (1969)
B ₂		80	
C	River Welland	dense	Proctor and Barden (1969)
D ₁	Monterey No. C	27	Lade and Duncan (1973)
D ₂		98	
E ₁	Ham River	loose	Reades and Green (1974)
E ₂		dense	

Figure 1.16. Effect of intermediate principal stress on the drained peak friction angle of several sands (Ladd et al. 1977).

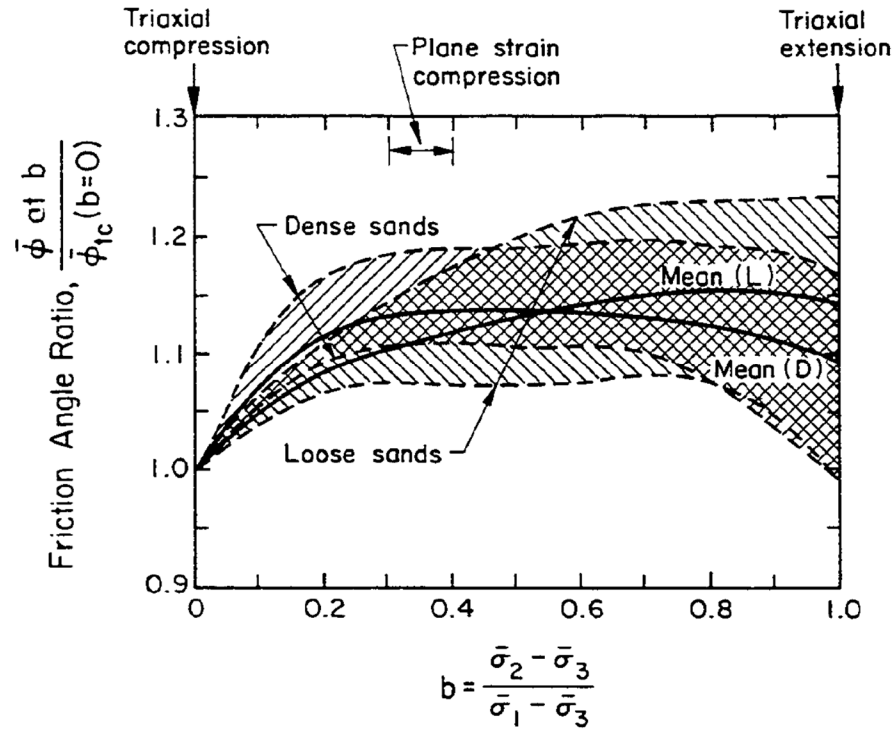


Figure 1.17. Effect of intermediate principal stress on the drained peak friction angle relative to that for triaxial compression (Kulhawy and Mayne 1990; data from Ladd et al. 1977).

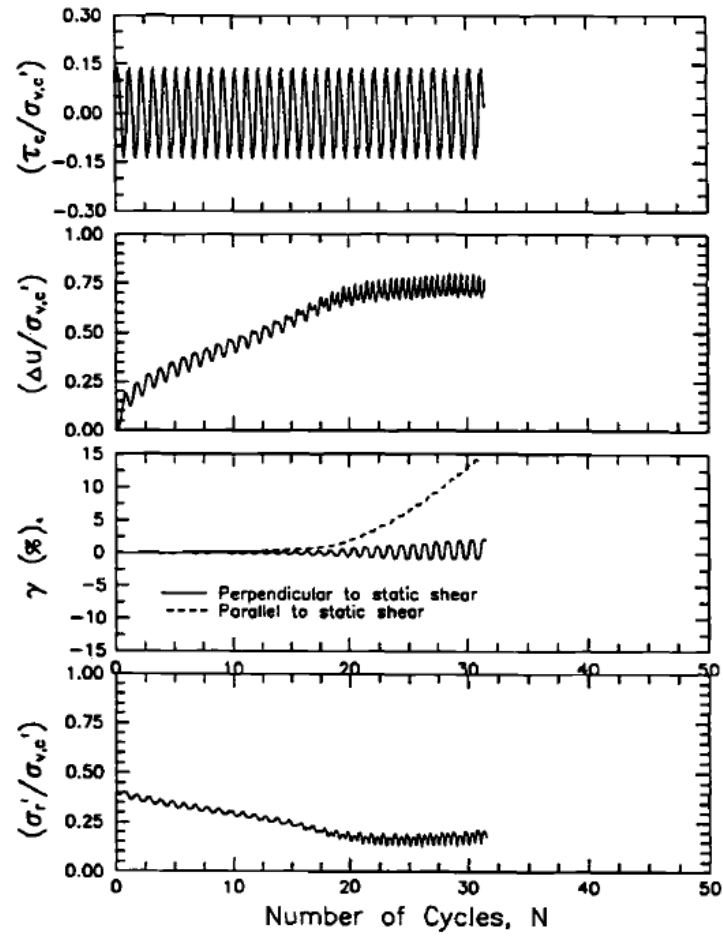


Figure 1.18. Undrained bidirectional cyclic DSS loading with an applied shear stress ($\alpha = 0.1$) perpendicular to the cyclic load for a sand at 55% relative density (Boulanger et al. 1993).

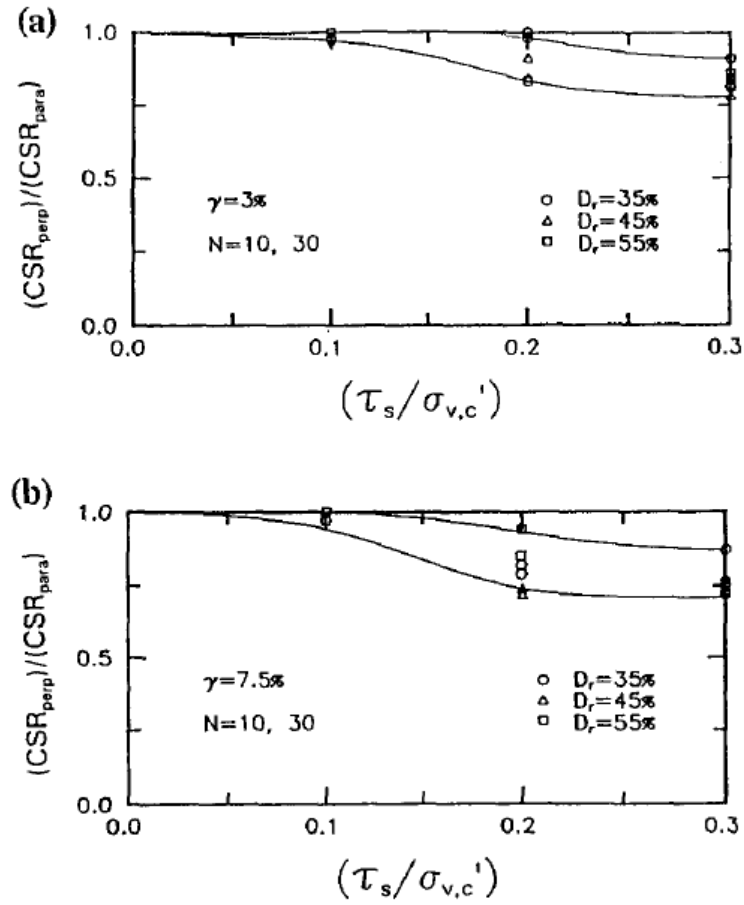


Figure 1.19. Reduction in cyclic resistance with initial static shear stresses applied perpendicular versus parallel to cyclic loads in bidirectional direct simple shear tests (Boulanger and Seed 1995).

2. MODEL FORMULATION

The sand plasticity model presented herein follows the basic framework of the stress ratio-controlled, critical state-compatible, bounding surface plasticity model for sand presented by Dafalias and Manzari (2004). The Dafalias and Manzari (2004) model extended the previous work by Manzari and Dafalias (1997) by adding a fabric-dilatancy tensor quantity to capture the effect of fabric evolution during loading. The fabric-dilatancy tensor enables a macroscopically representation of microscopically observed changes in sand fabric during plastic dilation, particularly their impact on the subsequent contractive response upon reversal of loading direction. Dafalias and Manzari (2004) provide a detailed rationale for this modeling framework, beginning with a triaxial formulation for clarity, followed by its generalization to the multi-axial formulation. The model described herein is presented directly in its multi-axial formulation, alongside the original Dafalias-Manzari formulation for reference and comparison.

2.1 Basic stress and strain terms

The basic stress and strain terms for the model are as follows. The model is based on effective stresses, with the conventional prime symbol dropped from the stress terms for convenience because all stresses are effective for the model. The stresses are represented by the tensor $\boldsymbol{\sigma}$, the principal effective stresses σ_1 , σ_2 , and σ_3 , the mean effective stress p , the deviatoric stress tensor $\mathbf{s}$, and the deviatoric stress ratio tensor $\mathbf{r}$. The relationships between the various stress terms can be summarized as follows:

$$\boldsymbol{\sigma} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} \end{pmatrix} \quad (1)$$

$$p = \frac{\sigma_{xx} + \sigma_{yy} + \sigma_{zz}}{3} \quad (2)$$

$$\mathbf{s} = \boldsymbol{\sigma} - p\mathbf{I} = \begin{pmatrix} s_{xx} & s_{xy} & s_{xz} \\ s_{xy} & s_{yy} & s_{yz} \\ s_{xz} & s_{yz} & s_{zz} \end{pmatrix} = \begin{pmatrix} \sigma_{xx} - p & \sigma_{xy} & \sigma_{xz} \\ \sigma_{xy} & \sigma_{yy} - p & \sigma_{yz} \\ \sigma_{xz} & \sigma_{yz} & \sigma_{zz} - p \end{pmatrix} \quad (3)$$

$$\mathbf{r} = \frac{\mathbf{s}}{p} = \begin{pmatrix} r_{xx} & r_{xy} & r_{xz} \\ r_{xy} & r_{yy} & r_{yz} \\ r_{xz} & r_{yz} & r_{zz} \end{pmatrix} = \begin{pmatrix} \frac{\sigma_{xx} - p}{p} & \frac{\sigma_{xy}}{p} & \frac{\sigma_{xz}}{p} \\ \frac{\sigma_{xy}}{p} & \frac{\sigma_{yy} - p}{p} & \frac{\sigma_{yz}}{p} \\ \frac{\sigma_{xz}}{p} & \frac{\sigma_{yz}}{p} & \frac{\sigma_{zz} - p}{p} \end{pmatrix} \quad (4)$$

Note that $\mathbf{I}$ is the identity matrix. In addition, the deviatoric stress and deviatoric stress ratio tensors are symmetric and have a zero trace such that $s_{xx} + s_{yy} + s_{zz} = 0$.

The model strains are represented by a tensor $\boldsymbol{\varepsilon}$, which can be separated into the volumetric strain ε_v and the deviatoric strain tensor $\boldsymbol{e}$. The volumetric strain, under a small-strain assumption, is,

$$\varepsilon_v = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz} \quad (5)$$

and the deviatoric strain tensor is,

$$\boldsymbol{e} = \boldsymbol{\varepsilon} - \frac{\varepsilon_v}{3} \mathbf{I} = \begin{pmatrix} \varepsilon_{xx} - \frac{\varepsilon_v}{3} & \varepsilon_{xy} & \varepsilon_{xz} \\ \varepsilon_{xy} & \varepsilon_{yy} - \frac{\varepsilon_v}{3} & \varepsilon_{yz} \\ \varepsilon_{xz} & \varepsilon_{yz} & \varepsilon_{zz} - \frac{\varepsilon_v}{3} \end{pmatrix} \quad (6)$$

In incremental form, the deviatoric and volumetric strain terms are decomposed into an elastic and a plastic part,

$$d\boldsymbol{e} = d\boldsymbol{e}^{el} + d\boldsymbol{e}^{pl} \quad (7)$$

$$d\varepsilon_v = d\varepsilon_v^{el} + d\varepsilon_v^{pl} \quad (8)$$

where,

$d\boldsymbol{e}^{el}$ = elastic deviatoric strain increment tensor

$d\boldsymbol{e}^{pl}$ = plastic deviatoric strain increment tensor

$d\varepsilon_v^{el}$ = elastic volumetric strain increment

$d\varepsilon_v^{pl}$ = plastic volumetric strain increment

2.2 Critical state

Dafalias and Manzari (2004), based on findings in Li and Wang (1998), used a power relationship to approximate the curving of the critical state line (Schofield and Wroth 1968) that occurs over a broad range of confining stresses,

$$e_{cs} = e_o - \lambda \left(\frac{p_{cs}}{p_A} \right)^m \quad (9)$$

where p_{cs} = mean stress at critical state, e_{cs} = critical state void ratio, and e_o , λ , and m are parameters controlling the position and shape of the critical state line. The state of the sand was then described

using the state parameter (Been and Jefferies 1985), which is the difference between the current void ratio (e) and the critical state void ratio (e_{cs}) at the same mean effective stress (p).

The model presented herein instead uses the relative state parameter index (ξ_R) as presented in Boulanger (2003a) and shown in Figure 2.1(a). The relative state parameter (Konrad 1988) is the state parameter normalized by the difference between the maximum void ratio (e_{max}) and minimum void ratio (e_{min}) values that are used to define relative density (D_R). The relative state parameter "index" is just the relative state parameter defined using an empirical relationship for the critical state line. Boulanger (2003a) used Bolton's (1986) dilatancy relationship to define the empirical critical state line and thus arrived at,

$$\xi_R = D_{R,cs} - D_R \quad (10)$$

$$D_{R,cs} = \frac{R}{Q - \ln\left(100 \frac{p}{p_A}\right)} \quad (11)$$

where $D_{R,cs}$ = relative density at critical state for the current mean effective stress (p). The parameters Q and R were shown by Bolton (1986) to be about 10 and 1.0, respectively, for quartzitic sands. Critical state lines using the above expression with Q values of 9 and 10 with R values of 1.0 and 1.5 are shown in Figure 2.1(b).

2.3 Bounding, dilatancy, and critical surfaces

The model incorporates critical, bounding, and dilatancy surfaces following the formulation of Dafalias and Manzari (2004). The present model includes the Lode angle dependency. Therefore, friction angles vary depending on loading path (e.g., compression or extension loading).

For the present implementation, the critical state surface is calculated to be consistent with the constant volume or critical state effective friction angle, interpreted as measured in a triaxial compression test.

$$M = \frac{6 \cdot \sin(\phi_{cv,tc})}{3 - \sin(\phi_{cv,tc})} \quad (12)$$

where $\phi_{cv,tc}$ is the critical state effective friction angle in triaxial compression.

The bounding and dilatancy surfaces are related to the critical surface as follows,

$$M^b = M \cdot g \cdot \exp(-n^b \xi_R) \quad (13)$$

$$M^d = M \cdot g \cdot \exp(n^d \xi_R) \quad (14)$$

where g defines the Lode angle dependence, and n^b and n^d are parameters determining the values of M^b and M^d , respectively. The three surfaces can be visualized as cones, shown in Figure 2.2 in principal stress space (σ_1 , σ_2 , and σ_3 axes) or as rounded triangles using principal stress-ratio space (r_1 , r_2 , and r_3 axes) as shown in Figure 2.3. The critical state surface in these figures is shown at a value of $g \cdot M$.

The current stress ratio M_{cur} can be computed as,

$$M_{cur} = \sqrt{\frac{3}{2}(\mathbf{r} : \mathbf{r})} \quad (15)$$

As the model is sheared toward critical state ($\xi_R = 0$), the values of M^b and M^d will both approach the value of $g \cdot M$. Thus, the bounding and dilatancy surfaces move together during shearing until they coincide with the critical state surface when the soil has reached critical state.

The above functional form for the bounding stress ratio controls the relationship between peak friction angle and relative state, which is consistent with the forms and data previously proposed by Been and Jefferies (1985) and Konrad (1986). The data from those studies were primarily for sands that were dense of critical, and the above relationship can reasonably fit those data. The few data points for loose-of-critical sands show that the peak friction angles (presumably determined at the limit of strains possible within the laboratory tests) were only slightly smaller than the critical state values, such that extending the above relationships to loose-of-critical sands may tend to underestimate the peak friction angles. Consequently, the present model allows n^b and n^d to be different for loose-of-critical and dense-of-critical states for the same sand.

The same factor g is applied to the bounding and dilation surfaces, as has commonly been assumed for convenience for these types of models (e.g., Manzari and Dafalias 1997, Zhang et al. 2023). However, Kutter et al. (1994) observed that the bounding and dilatancy surfaces may not be concentric with each other, which is an attribute considered in the calibration of the Lode angle dependency discussed below.

Lode angle dependency

The Lode angle dependency allows soil strengths to vary with loading paths, such as triaxial compression, direct simple shear, or triaxial extension. The Lode angle's effect on soil strength is incorporated using a factor g applied to the bounding and dilatancy surfaces. The form of the equation for g shown below is from Loukidis and Salgado (2009),

$$g(\theta, c) = \frac{2^{n_s} c}{\left[\left(I + c^{1/n_s} \right) - \left(I - c^{1/n_s} \right) \cos(3\theta) \right]^{n_s}} \quad (16)$$

where c defines the ratio of the soil's strength in triaxial extension to the soil's strength in triaxial compression, θ is the Lode angle of the current loading path, and n_s is a calibration parameter. They

noted that setting $n_s = 1$ returns the equation used by Manzari and Dafalias (1997) and setting $n_s = 0.25$ returns the equation used by Sheng et al. (2000). This equation with $n_s = 0.25$ was also used for the SANISAND-H model by Zhang et al. (2023). Loukidis and Salgado (2009) showed the range of c and n_s combinations that maintain bounding surface convexity in the deviatoric plane, which is important for numerical stability during integration. Their results showed that bounding surface convexity is maintained with c greater than about 0.64 for $n_s = 0.25$. The present model adopts $n_s = 0.25$.

During loading, the Lode angle is calculated at each step along the loading path based on the tensor $\mathbf{n}$. Note that θ is calculated in terms of its cosine, as used above, and varies from 0 in triaxial compression to $\pi/3$ in triaxial extension. The tensor $\mathbf{n}$ is calculated using the current stress ratio and the back-stress ratio tensor ($\boldsymbol{\alpha}$). The back-stress ratio tensor is discussed further in the following section.

$$\cos(3\theta) = \sqrt{6} \cdot \text{tr}(\mathbf{n}^3) \quad (17)$$

$$\mathbf{n} = \frac{\mathbf{r} - \boldsymbol{\alpha}}{|\mathbf{r} - \boldsymbol{\alpha}|} \quad (18)$$

The factor g determines the effect that intermediate principal stress has on peak friction angle. Note that θ is algebraically related to the intermediate principal stress ratio b ,

$$b = \frac{\sigma_2 - \sigma_3}{\sigma_1 - \sigma_3} \quad (19)$$

such that the effect of intermediate principal stress on peak friction angle can be expressed as a function of b (e.g., Figure 1.16 and Figure 1.17) or θ (e.g., Figure 2.4). Measuring the variation of friction angle with b or θ is challenging, as illustrated by the experimental results in Figure 2.4 (Lade et al. 2008) showing significant differences in the peak friction angles measured in true triaxial and torsional shear tests at larger b values. Further, peak friction angles are affected by fabric anisotropy, as illustrated in Figure 2.5 and Figure 2.6, which is difficult to quantify in practice. Despite these challenges, the experimental results are reasonably consistent in showing peak ϕ for drained plane-strain conditions are generally about 8-18% greater than for drained triaxial compression.

The variation of peak friction angle with b or θ , given the adopted equation for g , is determined by the parameter c . Note that peak friction angles are computed from the major and minor principal stresses as,

$$\phi = 2 \left(\arctan \left(\sqrt{\frac{\sigma_1}{\sigma_3}} \right) - 45^\circ \right) \quad (20)$$

The principal stresses on the bounding surface are defined by the values of M^b and θ (or b), from which the corresponding ϕ can be computed. The resulting variation of ϕ with b is illustrated in Figure 2.7 for $c = 0.71$ and peak friction angles in triaxial compression ($\phi_{b=0}$) of 30, 35, 40, and 45 degrees. These

results show that using $c = 0.71$ results in reasonable ratios for $\phi/\phi_{b=0}$ for $\phi_{b=0} = 30$ or 35 degrees but yields larger than desired ratios for $\phi_{b=0} = 40$ or 45 degrees. The effect of varying c is illustrated in Figure 2.8 showing ϕ and $\phi/\phi_{b=0}$ results for c values of 0.66 , 0.71 , and 0.76 with $\phi_{b=0} = 30$ and 40 degrees. These results show that $\phi/\phi_{b=0}$ ratios can be kept in reasonable ranges by adjusting the parameter c . Note that a fixed c means that $\phi/\phi_{b=0}$ ratios vary with $\phi_{b=0}$ which will also evolve during shearing in dilative sands; e.g., as ϕ decreases from its peak value toward its critical state value during drained shearing.

The default calibration for c for the present model expresses c as a function of $(M^b)_{b=0}$, which evolves in response to changes in the relative state parameter index,

$$c = 0.91 - 0.15(M^b)_{b=0} \geq 0.64 \quad (21)$$

This relationship was calibrated so that the evolving shape of the bounding surface during shearing maintains $\phi/\phi_{b=0}$ ratios that remain similar (for $\phi_{b=0}$ values up to 45 degrees) and are reasonably consistent with experimentally observed values. The resulting variation of ϕ and $\phi/\phi_{b=0}$ with b is illustrated in Figure 2.9, and compared to the empirical trends from Kulhawy and Mayne (1990) in Figure 2.10. This default calibration provides reasonable variations of ϕ with b , with the further advantage that its repeatability can simplify the calibration process for users.

2.4 Yield surface and image back-stress ratio tensors

The yield surface and back-stress ratio tensor follow those of the Dafalias-Manzari model. The yield surface is a small cone in stress space, and is defined in stress terms by the following expression:

$$f = \sqrt{(s - p\alpha):(s - p\alpha)} - \sqrt{\frac{2}{3}}pm = 0 \quad (22)$$

The back-stress ratio tensor α defines the center of the yield surface, and the parameter m defines the radius of the cone in terms of stress ratio. The yield function can be rewritten using stress ratio terms as follows,

$$f = \sqrt{(r - \alpha):(r - \alpha)} - \sqrt{\frac{2}{3}}m = 0 \quad (23)$$

The yield function can then be visualized as related to the distance between the stress ratio r and the back-stress ratio α , as illustrated in Figure 2.3.

The bounding surface formulation now requires that bounding and dilatancy stress ratio tensors be defined. Dafalias and Manzari (2004) showed that it is more convenient to track back-stress ratios and to similarly define bounding and dilatancy surfaces in terms of back-stress ratios. An image back-stress ratio tensor for the bounding surface (α^b) is defined as,

$$\alpha^b = \sqrt{\frac{2}{3}} [M^b - m] \mathbf{n} \quad (24)$$

where the tensor $\mathbf{n}$ is defined in the previous section. An image back-stress ratio tensor for the dilatancy surface (α^d) is similarly defined as,

$$\alpha^d = \sqrt{\frac{2}{3}} [M^d - m] \mathbf{n} \quad (25)$$

The computation of constitutive responses can now be more conveniently expressed in terms of back-stress ratios rather than in terms of stress ratios, as noted by Dafalias and Manzari (2004). These terms are included in Figure 2.3.

2.5 Stress reversal and initial back-stress ratio tensors

The bounding surface formulation, as described in Dafalias (1986) and adopted by Dafalias and Manzari (2004), keeps track of the initial back-stress ratio (α_{in}) and uses it in the computation of the plastic modulus K_p . This tracking of one instance in loading history is essentially a first-order method for tracking loading history. A reversal in loading direction is then identified, following traditional bounding surface practice, whenever

$$(\alpha - \alpha_{in}) : \mathbf{n} < 0 \quad (26)$$

A reversal causes the current stress ratio to become the initial stress ratio for subsequent loading. Small cycles of load reversal can reset the initial stress ratio and cause the plastic modulus K_p to increase accordingly, in which case the stress-strain response becomes overly stiff after a small load reversal. This is a well-known problem in bounding surface formulations for which various approaches offer different advantages and disadvantages.

The model presented herein tracks an initial back-stress ratio and a previous initial back-stress ratio (α_{in}^p), as illustrated in Figure 2.11. When a reversal occurs, the previous initial back-stress ratio is updated to the initial back-stress ratio, and the initial back-stress ratio is updated to the current back-stress ratio.

In addition, the model tracks an apparent initial back-stress ratio tensor (α_{in}^{app}) as schematically illustrated in Figure 2.11. The schematics in Figure 2.11(b) is similar to those of Figure 2.11(a), except that the most recent loading reversals correspond to a small unload-reload cycle on an otherwise positive loading branch. The components of α_{in}^{app} are taken as: (i) for positive loading directions, the minimum value they have ever had, but no smaller than zero, and (ii) for negative loading directions, the maximum value they have ever had, but no greater than zero. The use of α_{in}^{app} helps avoid the over-stiffening of the stress-strain response following small unload-reload cycles along an otherwise monotonically increasing branch of loading, without having to track the loading history through many cycles of load reversals.

The computation of K_p utilizes the values of α_{in}^{app} , α_{in}^{true} , and α_{in}^p , as defined in Figure 2.11, to better approximate the stress-strain response during an unload-reload cycle. For the last positive loading branch in this figure, the value of K_p is initially most strongly controlled (inversely) by the distance $(\alpha - \alpha_{in}^{true}):n$, such that the stiffness is initially large. As positive loading continues, the progressive reduction in K_p becomes increasingly dependent on α_{in}^{app} as well. Once the positive loading exceeds the previous reversal point, the value of K_p becomes solely dependent on the distance $(\alpha - \alpha_{in}^{app}):n$. Thus, the computation of K_p has the following dependencies:

$$\begin{aligned} \text{if } (\alpha - \alpha_{in}^p):n < 0 &\Rightarrow K_p = f(\alpha_{in}^{true}, \alpha_{in}^{app}) \\ \text{else} &\Rightarrow K_p = f(\alpha_{in}^{app}) \end{aligned} \quad (27)$$

The equations relating K_p to these back-stress ratios are given later in Section 2.7.

The impact of the above logic for defining α_{in} on stress-strain responses is demonstrated in Figure 2.12 showing α_{yy} versus axial strain ε_{yy} computed for two different drained triaxial loading simulations. For these two examples, the reloading stiffness of the current loading branch (green line) is initially large because K_p is initially computed based on $\alpha_{in} = \alpha_{in}^{true}$. As the loading exceeds α_{in}^p , the loading stiffness becomes much softer because K_p is now computed based on $\alpha_{in} = \alpha_{in}^{app}$.

2.6 Elastic strains and moduli

The elastic deviatoric strain and elastic volumetric strain increments are computed as:

$$d\epsilon^{el} = \frac{ds}{2G} \quad (28)$$

$$d\epsilon_v^{el} = \frac{dp}{K} \quad (29)$$

where G is the elastic shear modulus and K is the elastic bulk modulus. The elastic shear modulus in the model presented herein is dependent on the mean effective stress according to,

$$G = G_o p_A \left(\frac{p}{p_A} \right)^{1/2} C_{SR} \quad (30)$$

where G_o is a constant named the shear modulus coefficient, p_A is the atmospheric pressure (101.3 kPa), and C_{SR} is factor that accounts for stress ratio effects (described below).

Dafalias and Manzari (2004) had included dependence of G on void ratio following the form of Richart et al. (1970). This aspect was not included in the model herein because: (1) the effects of void ratio changes on G are small relative to those of confining stress, (2) the value of G_o is more strongly

affected by environmental factors such as cementation and ageing, and (3) the calibration of G to in-situ shear wave velocity data requires only one constant (G_o) rather than two (G_o and e).

Yu and Richart (1984) showed that the small-strain elastic shear modulus of sand is dependent on the principal stress ratio and principal stress ratio history. The effect of principal stress ratio was shown to generally be less than about 10% when the ratio of major to minor principal effective stresses is less than about 2.5, but to also increase to about 20-30% at higher principal stress ratios. They also showed that stress ratio history caused a reduction in the small-strain elastic shear modulus when the maximum previous principal stress ratio was greater than the current stress ratio. The effect of stress ratio and stress ratio history on the elastic shear modulus was approximately accounted for in the present model by the factor C_{SR} . The following equation for C_{SR} is similar in form to that used by Yu and Richart (1984) to represent principal stress ratio effects, except that it uses stress ratio terms consistent with the present model. The equation for C_{SR} is further normalized to produce a value of unity at the time of initialization and restricted to values less than unity thereafter, as,

$$C_{SR} = \frac{1 - C_{SR,o} \cdot \left(\frac{M}{M^b} \right)^{m_{SR}}}{\left| 1 - C_{SR,o} \cdot \left(\frac{M}{M^b} \right)^{m_{SR}} \right|_{initial}} \leq 1.0 \quad (31)$$

The above equation is for dense of critical states. For loose of critical state conditions, the M^b term in the numerator and denominator are replaced with M^d .

The above equation approximates Yu and Richart's (1984) results for stress ratio effects when $C_{SR,0} = 0.3$ and $m_{SR} = 2$. The effects of stress ratio history would cause further reductions and are more complicated to represent. The calibration examples presented later in this report worked well with $C_{SR,0} = 0.5$ and $m_{SR} = 4$, which keep the effect of stress ratio on elastic modulus small at small stress ratios, but lets the effect increase to a 60% reduction when the stress ratio is on the bounding surface.

The elastic bulk modulus is related to the shear modulus through the Poisson's ratio such that,

$$K = \frac{2(1+\nu)}{3(1-2\nu)} G \quad (32)$$

as was done by Dafalias and Manzari (2004).

2.7 Plastic components without fabric effects

Loading index

The loading index (L) is used to compute the plastic component of the volumetric strain increment and the plastic deviatoric strain increment tensor as,

$$d\epsilon_v^{pl} = \langle L \rangle D \quad (33)$$

$$de^{pl} = \langle L \rangle \mathbf{R}' \quad (34)$$

where D is the dilatancy, $\mathbf{R}$ is the direction of $d\epsilon_v^{pl}$, $\mathbf{R}'$ is the deviatoric component of $\mathbf{R}$, and $\langle \rangle$ are Macaulay brackets that set negative values to zero [i.e., $\langle L \rangle = L$ if $L \geq 0$, and $\langle L \rangle = 0$ if $L < 0$]. The tensor $\mathbf{R}$, as derived in Dafalias and Manzari 2004, is,

$$\mathbf{R} = \mathbf{R}' + \frac{1}{3} D \mathbf{I} = B \mathbf{n} - C \left(\mathbf{n}^2 - \frac{1}{3} \mathbf{I} \right) + \frac{1}{3} D \mathbf{I} \quad (35)$$

where $\mathbf{n}$ is the unit normal to the yield surface (Figure 2.3) and B and C are factors that depend on the chosen Lode angle surface (Section 2.3). Note that in this general case, $\mathbf{R}' \neq \mathbf{n}$. The factors B and C were rederived based on the choice of g presented in Section 2.3.

$$B = 1 + \frac{3}{8} \frac{c^4 - 1}{c^4} g^4 \cos(3\theta) \quad (36)$$

$$C = \frac{3\sqrt{6}}{8} \frac{c^4 - 1}{c^4} g^4 \quad (37)$$

The dilatancy D relates the incremental plastic volumetric strain to the incremental plastic deviatoric strain,

$$D = \frac{d\epsilon_v^{pl}}{|de^{pl}|} \quad (38)$$

Under triaxial loading, the dilatancy D can be also related to the conventional engineering shear strain, as,

$$D = \frac{d\epsilon_v^{pl}}{\sqrt{\frac{2}{3}} |d\gamma^{pl}|} \quad (39)$$

The loading index, as derived in Dafalias and Manzari (2004) is,

$$L = \frac{1}{K_p} \frac{\partial f}{\partial \sigma} : d\sigma = \frac{1}{K_p} [\mathbf{n} : ds - \mathbf{n} : r dp] \quad (40)$$

$$L = \frac{2G\mathbf{n} : d\mathbf{e} - \mathbf{n} : r K d\varepsilon_v}{K_p + 2G(B - C \text{tr}(\mathbf{n}^3) - K D \mathbf{n} : \mathbf{r})}$$

The stress increment for an imposed strain increment can then be computed as,

$$d\sigma = 2G d\mathbf{e} + K d\varepsilon_v \mathbf{I} - \langle L \rangle \left(2G \left[B \mathbf{n} - C \left(\mathbf{n}^2 - \frac{1}{3} \mathbf{I} \right) \right] + K D \mathbf{I} \right) \quad (41)$$

Hardening and the update of the back-stress ratio

Updating of the back-stress ratio is dependent on the hardening aspects of the model. Dafalias and Manzari (2004) updated the back-stress ratio according to bounding surface practice as,

$$d\alpha = \langle L \rangle \left(\frac{2}{3} \right) h (\alpha^b - \alpha) \quad (42)$$

where h is the hardening coefficient. The factor of 2/3 was included for convenience so that model constants would be the same in triaxial and multi-axial derivations. They subsequently showed that the consistency condition $\delta f = 0$ was satisfied when the plastic modulus K_p was related to the hardening coefficient as,

$$K_p = \frac{2}{3} p \cdot h \cdot (\alpha^b - \alpha) : \mathbf{n} \quad (43)$$

This expression can be rearranged to show that the consistency equation can be satisfied by expressing the hardening coefficient as,

$$h = \frac{3}{2} \cdot \frac{K_p}{p \cdot (\alpha^b - \alpha) : \mathbf{n}} \quad (44)$$

The relationship for the plastic modulus can subsequently take a range of forms, provided that the hardening coefficient and updating of the back-stress ratio follow the above expressions.

Plastic modulus

The plastic modulus in the multi-axial generalized form of Dafalias and Manzari (2004), after substituting in their expression for the hardening coefficient, can be expressed as,

$$K_p = \frac{2}{3} G \cdot h_o \cdot \left[\frac{1+e}{(2.97-e)^2} \cdot (1-C_h e) \right] \cdot \frac{(\alpha^b - \alpha):n}{(\alpha - \alpha_{in}):n} \quad (45)$$

where h_o and C_h are scalar parameters and e is the void ratio. Setting aside the secondary influence of void ratio, this form illustrates that K_p is proportional to G , proportional to the distance of the back-stress ratio to the bounding back-stress ratio, and inversely proportional to the distance of the back-stress ratio from the initial back-stress ratio.

The plastic modulus relationship was revised in the PM4Sand model to provide an improved approximation of empirical relationships for secant shear modulus and equivalent damping ratios during drained strain-controlled cyclic loading. The plastic modulus is computed as,

$$K_p = G \cdot h_o \cdot g^{-n_g} \frac{\left[(\alpha^b - \alpha):n \right]^{0.5}}{\left[\exp \left[(\alpha - \alpha_{in}^{app}):n \right] - 1 \right] + C_{\gamma 1}} C_{rev} \quad (46)$$

$$C_{rev} = \frac{(\alpha - \alpha_{in}^{app}):n}{(\alpha - \alpha_{in}^{true}):n} \quad \text{for } (\alpha - \alpha_{in}^p):n \leq 0$$

$$= 1 \quad \text{otherwise} \quad (47)$$

The factor C_{rev} accounts for the effect of unload-reload cycles as discussed in Section 2.5 and illustrated in Figure 2.12. The constant $C_{\gamma 1}$ in the denominator serves to avoid division by zero and has a slight effect on the nonlinearity and damping at small shear strains. If $C_{\gamma 1} = 0$, then the value of K_p will be infinite at the start of a loading cycle because $(\alpha - \alpha_{in}):n$ will also be zero. In that case, nonlinearity will become noticeable only after $(\alpha - \alpha_{in}):n$ becomes large enough to reduce K_p closer to the value of G (e.g., K_p/G closer to 100 or 200). Setting the value of $C_{\gamma 1} = h_o/200$ produces a reasonable response as will be demonstrated later with examples of modulus reduction and equivalent damping ratios. For stress ratios outside the bounding surface [i.e., loose-of-critical states with $(\alpha^b - \alpha):n < 0$], the plastic modulus is set to zero rather than allowing for negative values. This restriction on the plastic modulus improved numerical stability while having little effect on computed stress-strain responses. The plastic modulus is further modified for the effects of fabric and fabric history, as described in a later section.

The plastic modulus is further modified using the factor g^{-n_g} as previously implemented by Yang et al. (2020) within the SANISAND family of models. This factor allows control over the asymmetrical strain accumulation that occurs in triaxial loading based on differences due to Lode angle dependence. The same modifier is also used with dilatancy, as discussed in the following sections.

Plastic volumetric strains – Dilation

Plastic volumetric strains are related to plastic deviatoric strains through the dilatancy D (Equations 38 and 39 above). Dilatancy is computed herein following Dafalias and Manzari (2004) with the modification by Yang et al. (2020) (and with additional fabric effects described in a later section) as,

$$D = A_{do} \cdot g^{-n_g} \left[(\boldsymbol{\alpha}^d - \boldsymbol{\alpha}) : \mathbf{n} \right] \quad (48)$$

Note that dilation (increasing void ratio) occurs whenever the term $(\boldsymbol{\alpha}^d - \boldsymbol{\alpha}) : \mathbf{n}$ is less than zero whereas contraction (decreasing void ratio) occurs when it is positive.

The addition of a dependence on Lode angle to dilatancy yields decreased dilatancy in triaxial extension or direct simple shear versus in triaxial compression. Therefore, in undrained cyclic loading, the bias in strain towards triaxial extension or compression can be controlled using the parameter n_g .

The constant A_{do} in this relationship can be related to the dilatancy relationship proposed by Bolton (1986), which follows from the work of Rowe (1962), through the following sequence of steps. Bolton showed that the difference between peak and constant volume friction angles could be approximated in plane strain as,

$$\phi_{pk} - \phi_{cv} = -0.8\psi = 5I_R \quad (49)$$

where,

$$\psi = \tan^{-1} \left(\frac{d\varepsilon_v^{pl}}{|d\gamma^{pl}|} \right) \quad (50)$$

and in triaxial loading as,

$$\phi_{pk} - \phi_{cv} = 3I_R \quad (51)$$

Since $\psi \approx \tan(\psi)$ for ψ less than about 0.35 radians (20 degrees), the difference between peak and constant volume friction angles (in radians) in triaxial loading can be approximated as,

$$\phi_{pk} - \phi_{cv} = -0.48\psi \quad (52)$$

Using Equations 39 and 51, ψ can be defined in terms of dilatancy, D , in triaxial loading as,

$$\psi = \sqrt{\frac{2}{3}}D \quad (53)$$

The peak friction angle is mobilized at the bounding surface, so this can be written as,

$$\begin{aligned} \phi_{pk} - \phi_{cv} &= -0.48\sqrt{\frac{2}{3}}A_{do} \cdot [(\alpha^d - \alpha) : \mathbf{n}] \\ \phi_{pk} - \phi_{cv} &= -0.48\sqrt{\frac{2}{3}}A_{do} \cdot \left[\left(\sqrt{\frac{2}{3}}g(M^d \mathbf{n} - M^b \mathbf{n}) \right) : \mathbf{n} \right] \end{aligned} \quad (54)$$

The term $\mathbf{n}:\mathbf{n}$ is equal to unity, and the values of ϕ_{pk} and ϕ_{cv} (again in radians) can be replaced with expressions in terms of M^b and M , once again assuming triaxial loading, as,

$$\sin^{-1}\left(\frac{3M^b}{6+M^b}\right) - \sin^{-1}\left(\frac{3M}{6+M}\right) = \frac{2}{3}0.48A_{do} \cdot [M^b - M^d] \quad (55)$$

This expression can then be rearranged to solve for A_{do} under the assumption of triaxial loading as,

$$A_{do} = \frac{\sin^{-1}\left(\frac{3M^b}{6+M^b}\right) - \sin^{-1}\left(\frac{3M}{6+M}\right)}{\frac{2}{3}0.48(M^b - M^d)} \quad (56)$$

where the angles returned by the $\sin^{-1}$ functions are in radians.

The parameter A_{do} should thus be chosen to be consistent with the n^d and n^b terms that control M^b , and M^d . For example, setting the parameters n^b and n^d equal to 0.5 and 0.1, respectively, results in A_{do} varying from 1.04 for $\xi_R = -0.1$ to 1.09 for $\xi_R = -0.7$. A default value for A_{do} is computed based on the above expression using the conditions at the time of model initialization in FLAC3D (as described in a later section). If an alternative value for A_{do} is manually input as a property of the model, then the default value will be deactivated.

Effect of fabric on peak mobilized friction angles in drained and undrained loading

Kutter and Chen (1997) showed that plastic dilation rates are different in drained and undrained loading of the same clean sand, with the consequence being that the peak mobilized friction angles are also different for drained and undrained loading. This aspect of behavior would appear to be contradictory to having a bounding surface that is only dependent on the relative state of the sand (i.e., through the parameter n^b) if the mobilized friction angles for drained and undrained loading paths are both controlled by the bounding surface. The model presented herein produces the same peak mobilized

friction angles for drained and undrained loading because both conditions become limited by the same bounding surface.

Kutter et al. (1994) observed that the bounding and dilatancy surfaces may not be concentric with each other. This is contradictory to the current bounding surface formulation in which both surfaces are controlled by the same shape parameter c .

The above aspects of behavior deserve closer examination in future efforts.

Plastic volumetric strains – Contraction

Plastic volumetric strains during contraction (i.e., whenever $(\alpha^d - \alpha):n$ is greater than zero) are computed in the Dafalias and Manzari (2004) model using the same expression as used for dilatancy,

$$D = A_{do} \cdot \left[(\alpha^d - \alpha) : n \right] \quad (57)$$

Use of this expression limited the ability of the model to approximate several important loading responses; e.g., it greatly overestimated the slope of the cyclic resistance ratio (CRR) versus number of equivalent uniform loading cycles for undrained cyclic element tests (e.g., Figure 1.4).

Plastic volumetric strains during contraction for the model presented herein are computed using the following expression,

$$D = A_{dc} \cdot \left[(\alpha - \alpha_{in}^{app}) : n + C_{in} \right]^2 \frac{(\alpha^d - \alpha) : n}{(\alpha^d - \alpha) : n + C_D} \quad (58)$$

$$A_{dc} = \frac{A_{do} \cdot g^{-n_g}}{h_p} \quad (59)$$

The various forms in the above relationships were developed to improve different aspects of the calibrated model's performance. The value of D was set proportional to the square of $((\alpha - \alpha_{in}) : n + C_{in})$ to improve the slope of the relationship between CRR and number of uniform loading cycles. The C_{in} term depends on fabric and is described in a later section along with other modifications to the above expression for the effects of fabric and fabric history. The inclusion of the term C_{in} improves the stress paths for undrained cyclic loading and the volumetric strain response during drained cyclic loading; inclusion of this constant enables some volumetric strain to develop early in the unloading from a point outside the dilatancy surface (as described later). The remaining terms on the right-hand side of the equation were chosen to be close to unity over most of the loading range, while ensuring that D smoothly goes to zero as α approaches α^d ; reasonable results were obtained using a C_D value of 0.10.

The parameter A_{dc} for contraction was related to the value of A_{do} for dilation by dividing it by a parameter h_p that can be varied during the calibration process to obtain desired cyclic resistance ratios. The effect of confining stress on cyclic loading behavior was then conveniently incorporated by making

h_p depend on ξ_R , with the following form chosen so that the model produces results consistent with the design K_σ relationships presented earlier in Figures 1.4 and 1.5:

$$h_p = h_{po} \exp\left(0.9 + 6.1(0.5 - \xi_R)^{2.1}\right) \quad \text{for } \xi_R \leq 0.5 \quad (60)$$

$$h_p = h_{po} \exp(0.9) \quad \text{for } \xi_R > 0.5 \quad (61)$$

Thus, the scalar constant h_{po} provides a linear scaling of contraction rates while the functional form of the remaining portion in Equations (60) and (61) controls the effect of overburden stress on CRR. The variation of h_p with ξ_R for different values of h_{po} is plotted in Figure 2.13. Once the other input parameters have been selected, the constant h_{po} can be calibrated to arrive at a desired cyclic resistance ratio.

An upper limit was imposed on the contraction rate, with the limiting value computed as,

$$D \leq 1.5 \cdot A_{do} \frac{(\alpha^d - \alpha) : \mathbf{n}}{(\alpha^d - \alpha) : \mathbf{n} + C_D} \quad (62)$$

This limit prevented numerical issues that were encountered with excessively large contraction rates. It does not appear to have limited the ability of the model to recreate realistic contraction rates as illustrated in the calibration examples shown later.

2.8 Fabric Effects

Dafalias and Manzari (2004) introduced a fabric-dilatancy tensor ($\mathbf{z}$) that could be used to account for the effects of prior straining. Their fabric tensor ($\mathbf{z}$) evolved in response to plastic volumetric dilation strains, according to,

$$d\mathbf{z} = -c_z \langle -d\varepsilon_v^{pl} \rangle (z_{max} \mathbf{n} + \mathbf{z}) \quad (63)$$

where the parameter c_z controls the rate of evolution and z_{max} is the maximum value that $\mathbf{z}$ can attain.

The fabric-dilatancy tensor was modified for the present model as,

$$d\mathbf{z} = -\frac{c_z}{1 + \left\langle \frac{z_{cum}}{2\sqrt{2}z_{max}} - 1 \right\rangle} \frac{\langle -d\varepsilon_v^{pl} \rangle}{D} (z_{max} \mathbf{n} + \mathbf{z}) \quad (64)$$

In this expression, the tensor $\mathbf{z}$ evolves in response to plastic deviatoric strains that occur during dilation only (i.e., dividing the plastic volumetric strain by the dilatancy gives plastic shear strain). In addition, the evolution of fabric is restricted to only occur when $(\boldsymbol{\alpha}^d - \boldsymbol{\alpha}) : \mathbf{n} < 0$; this additional constraint precludes fabric evolution during dilation above the rotated dilatancy surface (introduced later) but below the non-rotated dilatancy surface. The parameter z_{cum} is the cumulative value of absolute changes in $\mathbf{z}$ computed according to,

$$dz_{cum} = |d\mathbf{z}| \quad (65)$$

The rate of evolution for $\mathbf{z}$ therefore decreases with increasing values of z_{cum} , which enables the undrained cyclic stress-strain response to progressively accumulate shear strains rather than lock-up into a repeating stress-strain loop. In addition, the greatest past peak value (scalar amplitude) for $\mathbf{z}$ during its loading history is also tracked,

$$z_{peak} = \max\left(\sqrt{\mathbf{z} : \mathbf{z}}, z_{peak}\right) \quad (66)$$

The values of $\mathbf{z}$, z_{peak} , and z_{cum} are later used to facilitate the accumulation of shear strains under symmetric loading through their effects on the plastic modulus and dilatancy relationships.

The evolution of the fabric tensor terms is illustrated in Figure 2.14 and Figure 2.15 showing the response of a medium dense sand to undrained cyclic DSS loading without any sustained horizontal shear stress (Figure 2.14) and with a sustained horizontal shear stress (Figure 2.15). Figure 2.16 showcases the response of sand at the same relative density (55%) to undrained cycle triaxial loading without any sustained deviatoric shear stress. These figures show the stress path and stress-strain response of the sand, along with time histories for the back-stress ratios and fabric tensor terms. Note how the fabric terms do not grow until the soil reaches the dilatancy surface, and how the stress-ratios are limited by the bounding stress ratio. There is no horizontal shear stress reversal for the case shown in Figure 2.15 and thus the back-stress ratio and fabric terms do not reverse either.

Additional memory of fabric formation history

Memory of the fabric formation history was included in the model presented herein to improve the ability of the model to account for the effects of sustained static shear stresses and account for differences in fabric effects for various drained versus undrained loading conditions.

The initial fabric tensor ($\mathbf{z}_{in}$) at the start of the current loading path is determined whenever a stress ratio reversal occurs, and thus corresponds to the same times that the initial back-stress ratio and previous initial back-stress ratio are updated. The $\mathbf{z}_{in}$ tracks the immediate history terms without any consideration of whether an earlier loading cycle had produced greater degrees of fabric (i.e., the logic is different from that adopted for the updating of back-stress ratio history terms). This history term is used for describing the degree of stress rotation and its effects on plastic modulus, as described later.

Another aspect of the fabric history that is tracked is the mean stress at which the fabric is formed. This aspect of fabric history is tracked via the product of $\mathbf{z}$ and $\mathbf{p}$, and defining p_{zp} as the mean stress at the time that this product achieves its greatest peak value. The p_{zp} is used in addressing a couple of issues, including the issue of how fabric that is formed during liquefaction may be erased during reconsolidation. For example, saturated sand that develops cyclic mobility behavior during undrained cyclic loading clearly remembers its history of plastic deviatoric strains and then subsequently forgets (to a large extent) this prior strain history when it reconsolidates back to its pre-earthquake confining stress. As another example, the memory of prior strains during undrained cyclic loading is very different than the memory of prior strains during drained cyclic loading. This memory conceptually could be related to the history of plastic and total volumetric strains, but a simpler method to account for this effect is to consider how the mean stress p relates to the value of p_{zp} . Conceptually, it appears that prior strain history (or fabric) is most strongly remembered when the soil is operating under mean stresses that are smaller than those that existed when the fabric was formed (i.e., $p \ll p_{zp}$) and then largely forgotten when they are of the same order (i.e., $p \approx p_{zp}$). This attribute will be used in the relationships described later for describing the effects of fabric on dilatancy.

Effect of fabric on plastic modulus

An effect of fabric on the plastic modulus was added to the model presented herein by reducing the plastic modulus as the fabric tensor increased in peak amplitude, as follows:

$$K_p = G \cdot h_o \cdot g^{-n_g} \frac{\left[(\mathbf{a}^b - \mathbf{a}) : \mathbf{n} \right]^{0.5}}{\left[\exp((\mathbf{a} - \mathbf{a}_{in}^{app}) : \mathbf{n}) - 1 \right] + C_{\gamma 1}} C_{rev} \cdot \frac{C_{K\alpha}}{1 + C_{K_p} \left(\frac{z_{peak}}{z_{max}} \right) \langle (\mathbf{a}^b - \mathbf{a}) : \mathbf{n} \rangle \sqrt{1 - C_{zpk2}}} \quad (67)$$

where,

$$C_{K\alpha} = 1 + \frac{C_{K\alpha f}}{1 + \left(2.5 \cdot \langle (\mathbf{a} - \mathbf{a}_{in}^{true}) : \mathbf{n} \rangle \right)^2} \cdot C_{pzp2} \cdot C_{zpk1} \quad (68)$$

$$C_{zpk1} = \frac{z_{peak}}{z_{cum} + \frac{z_{max}}{5}} \quad (69)$$

$$C_{zpk2} = \frac{z_{peak}}{z_{cum} + \frac{z_{max}}{100}} \quad (70)$$

$$C_{pzp2} = \frac{-\langle -(p_{zp} - p) \rangle}{-\langle -(p_{zp} - p) \rangle + p_{\min}} \quad (71)$$

The above expressions produce a reduction in plastic modulus when fabric is favorable ($\mathbf{z}:\mathbf{n} \geq 0$) and with increasing plastic shear strains (which conceptually would break down any cementation). This reduces both the plastic modulus and the hysteretic damping at larger shear strains (note that $z_{\text{peak}} = 0$ unless the soil has been loaded strongly enough to pass outside the dilatancy surface), improves the volumetric strains that develop in drained cyclic loading, and improves the path in undrained cyclic loading.

The $C_{K\alpha}$ and square root of $(1-C_{zpk2})$ terms both serve to increase K_p during non-reversal loading by amounts that depend on the fabric and stress history. During reversal loading, the $(1-C_{zpk2})$ term approaches unity and K_p evolves as it previously had. The roles of each of the other terms are discussed below.

C_{zpk1} and C_{zpk2} are evolution terms that initiate at zero and grow to be unity under monotonic (non-reversing) loading conditions, corresponding to unidirectional fabric development. These two terms differ in the rate at which they approach unity, which is controlled through the scaling constants $z_{\text{max}}/5$ and $z_{\text{max}}/100$, respectively. These values are selected to better reproduce the engineering behaviors of interest and the empirical correlations presented later. Under full reversal loading conditions, where the fabric alternates between positive and negative values, both C_{zpk1} and C_{zpk2} tend toward zero.

C_{pzp2} is initially zero and remains zero until fabric development occurs. After fabric is formed, this term rapidly transitions to unity for mean effective stress p values less than the stress level p at which maximum fabric was formed (p_{zp}). If p increases beyond the value of p_{zp} , C_{pzp2} returns to zero as enforced by the Macaulay brackets.

The values for the calibration parameters C_{Kp} and $C_{K\alpha f}$ were chosen for their ability to reasonably approximate the targeted behaviors, as discussed later. Setting C_{Kp} to a default value of $\sqrt{2}$ was found to produce reasonable responses with particular emphasis on improving (reducing) the equivalent damping ratios at shear strains of 1 to 3% in drained cyclic loading. The parameter $C_{K\alpha f}$ plays a key role in controlling the undrained cyclic loading response in the presence of sustained static shear stresses; a default calibration for $C_{K\alpha f}$, dependent on D_R is presented later.

The cumulative effect of the above parameters can be interpreted as follows. Under strong unidirectional loading, the soil develops significant fabric. Upon unloading and subsequent reloading, both C_{pzp2} and C_{zpk1} attain values near unity, resulting in an elevated $C_{K\alpha}$. If loading continues and the soil is sheared to stress levels exceeding those at which the previous maximum fabric was formed (i.e., at mean effective stresses $p > p_{zp}$), then $C_{pzp2} = 0$ and $C_{K\alpha}$ correspondingly returns to unity. In this manner, an element that has developed strong fabric under monotonic or cyclic loading without reversal of the shear stress direction (e.g., an element within a steep slope where static shear stresses dominate over cyclic components) exhibits an initially stiff response upon reloading (increased K_p). This is followed by softening (smaller K_p) as loading progresses into previously unexperienced (virgin) stress states.

Effect of fabric on plastic volumetric dilation

A rotated dilatancy surface with slope M^{dR} which evolves with the history of the fabric tensor $\mathbf{z}$ was added to the framework of the model to facilitate earlier dilation at low stress ratios under certain loading paths (Ziotopoulou 2014, Ziotopoulou and Boulanger 2016). The rotated surface, schematically illustrated in Figure 2.17 as a cone in principal stress space and Figure 2.18 as a rounded triangular surface on a principal stress-ratio plot, is equal to the original dilatancy surface scaled-down by a factor C_{rot1} :

$$M^{dR} = \frac{M^d}{C_{rot1}} \quad (72)$$

$$C_{rot1} = 1 + \frac{\sqrt{2} \cdot \langle -\mathbf{z} : \mathbf{n} \rangle}{z_{max}} \cdot (1 - C_{zin1}) \geq 1 \quad (73)$$

where M^d is the slope of the unrotated dilatancy surface. Experimental results (Ziotopoulou 2014) indicate that the loading history, the loading direction and the loading pattern play important roles in the response of the soil to irregular cyclic loading. Thus, the scaling factor that defines the rotated dilatancy surface was made dependent on whether fabric is favorable ($\mathbf{z} : \mathbf{n} > 0$) or unfavorable ($\mathbf{z} : \mathbf{n} < 0$) and on the factor C_{zin1} which is an indirect measure of whether there are reversals or not:

$$C_{zin1} = \left\langle 1 - \exp \left(-2.0 \left| \frac{\mathbf{z}_{in} : \mathbf{n} - \mathbf{z} : \mathbf{n}}{z_{max}} \right| \right) \right\rangle \quad (74)$$

where $\mathbf{z}_{in}$ is the fabric tensor at the beginning of the current loading branch. C_{zin1} can take values ranging from 0, when there are no reversals, to 1, when there are reversals. The rotated dilatancy surface is operating only for loading with an unfavorable fabric since the factor C_{rot1} becomes 1 when the fabric is favorable (i.e., $\langle -\mathbf{z} : \mathbf{n} \rangle = 0$).

A back-stress ratio tensor for the rotated dilatancy surface (α^{dR}) was introduced as:

$$\alpha^{dR} = \sqrt{\frac{2}{3}} \cdot (M^{dR} - m) \mathbf{n} \quad (75)$$

Dilation occurs whenever the term $(\alpha^{dR} - \alpha) : \mathbf{n}$ is negative whereas contraction occurs when it is positive. The calculation of D is still treated separately during dilation and contraction.

Dilatancy D during dilation is now computed according to the following expressions. First, a value for D is computed from the rotated dilatancy surface:

$$D_{rot} = A_d \cdot \frac{\langle -\mathbf{z} : \mathbf{n} \rangle}{z_{max}} \cdot \frac{(\boldsymbol{\alpha}^{dR} - \boldsymbol{\alpha}) : \mathbf{n}}{C_{DR}} \quad (76)$$

where the C_{DR} factor is applied to reduce the rate under which dilatancy is increasing and is discussed further below. Second, another value for D is computed that would be obtained from the non-rotated dilatancy surface:

$$D_{non-rot} = A_d \cdot \left(-\langle -(\boldsymbol{\alpha}^d - \boldsymbol{\alpha}) : \mathbf{n} \rangle \right) \quad (77)$$

The Macaulay brackets in the above expression ensure that $D_{non-rot}$ is equal to zero whenever $(\boldsymbol{\alpha}^d - \boldsymbol{\alpha}) : \mathbf{n} > 0$ while $(\boldsymbol{\alpha}^{dR} - \boldsymbol{\alpha}) : \mathbf{n} < 0$. Lastly, the operating value of D is selected from the above two values based on:

$$\begin{aligned} \text{if } D_{non-rot} < D_{rot} &\Rightarrow D = D_{non-rot} \\ \text{else } D &= D_{non-rot} + (D_{rot} - D_{non-rot}) \cdot \frac{\langle M^b - M^{cur} \rangle}{\langle M^b - M^{cur} + 0.01 \rangle} \end{aligned} \quad (78)$$

The above logic is illustrated in Figure 2.19, where the dilatancy D is plotted over a half loading cycle transitioning from contraction to dilation. The figure shows that $D_{non-rot}$ governs whenever it is smaller (more negative) than D_{rot} . When D_{rot} becomes smaller than $D_{non-rot}$, the value of D is obtained through interpolation using the additional term that scales the difference between D_{rot} and $D_{non-rot}$. This interpolation factor approaches unity for stress ratios away from the bounding surface ($M^{cur} < M^b$), such that D will be equal to D_{rot} as illustrated in the figure. However, as the stress ratio approaches the bounding surface, the interpolation factor smoothly decreases to zero, ensuring that dilatancy also goes to zero as the soil approaches the critical state ($M = M^d = M^b$). The constant of 0.01 in the denominator controls the rate of this transition and was found to provide reasonable results in trial simulations.

The factor C_{DR} in the denominator of the expression for D_{rot} is applied so that the D computed based on the rotated dilatancy surface is consistent with experimental observations. Its value, for the default calibration described later, has been made dependent on the initial D_R of the soil.

Lastly, the parameter A_d in the expressions for both D_{rot} and $D_{non-rot}$ is expressed as,

$$A_d = \frac{A_{do} (C_{zin2})}{\left(\frac{z_{cum}^2}{2z_{max}} \right) \left(1 - \frac{\langle -\mathbf{z} : \mathbf{n} \rangle}{z_{peak}} \right)^3 (C_\varepsilon)^2 (C_{pzp})(C_{pmin})(C_{zin1}) + 1} \quad (79)$$

$$C_{pzp} = \frac{1}{1 + \left(\frac{2.5p}{p_{zp}} \right)^5} \quad (80)$$

$$C_{pmin} = \frac{1}{1 + \left(\frac{p_{min2}}{p} \right)^2} \quad (81)$$

$$C_{zin1} = \left\langle 1 - \exp \left(-2.0 \left| \frac{\mathbf{z}_{in} : \mathbf{n} - \mathbf{z} : \mathbf{n}}{z_{max}} \right| \right) \right\rangle \quad (82)$$

$$C_{zin2} = \frac{1 + C_{zin1} \frac{z_{cum} - z_{peak}}{3z_{max}}}{1 + 3 \cdot C_{zin1} \frac{z_{cum} - z_{peak}}{3z_{max}}} \quad (83)$$

Consider the six terms added to the denominator of the expression for A_d . The first term $[z_{cum}^2/2z_{max}]$ facilitates the progressive growth of strains under symmetric loading by reducing the dilatancy that occurs when a liquefied soil has been sheared through many cycles of loading; note that this term progressively increases with subsequent cycles of loading. The second term facilitates strain-hardening when the plastic shear strain reaches the prior peak value, wherein the term approaches zero (i.e., when $\mathbf{z}:\mathbf{n}$ approaches z_{peak}) and the dilation rate consequently rapidly approaches the virgin loading value of A_{do} . The third term C_ε is a calibration constant that can be used to modify the rate of plastic shear strain accumulation. The fourth term C_{pzp} causes the effects of fabric on dilation to be diminished (erased) whenever the current value of p is near the value of p_{zp} ; this term enables the model to provide reasonable predictions of responses to large numbers of either drained or undrained loading cycles. The fifth term C_{pmin} provides a minimum amount of shear resistance for a soil after it has temporarily reached an excess pore pressure ratio of 100%; This term is almost zero when $p'=0$, such that the soil will initially dilate until some minimum p' has developed, after which the term quickly approaches 1.0. The parameter p_{min2} is currently set to become equal to 5% of the value of p' at consolidation (which is the value that exists when the flag FirstCall –see Section 3– was last set equal to 0), with the minimum value of p_{min2} being 10 times the minimum value of p' (i.e., $p_{min} = 1/200$ times the larger of p_A or the value of p' at consolidation). The sixth term C_{zin1} facilitates strain-hardening when stress reversals are not causing fabric changes; i.e., when the initial and current fabric terms are close to equal, the term C_{zin1} goes to zero. Lastly, the second term in the numerator, C_{zin2} , causes the dilatancy to be decreased by up to a factor of 3 under conditions of large strains and full stress (and fabric) reversals, which improves the prediction of cyclic strain accumulation during undrained cyclic loading.

An additional constraint is placed on D during dilation at very low effective stresses. For $p < 2p_{min}$, the value of D cannot be smaller in magnitude than computed by the following expression:

$$D = -3.5A_{do} \left\langle M^b - M^d \right\rangle \frac{2p_{min} - p}{p_{min}} \quad \text{for } p_{min} \leq p \leq 2p_{min} \quad (84)$$

This expression ensures that the model will, for dense of critical soils (i.e., $M^b > M^d$), be dilative when p falls below $2p_{\min}$.

Effect of fabric on plastic volumetric contraction

Dafalias and Manzari (2004) used the fabric tensor to modify the dilatancy during contraction ($D > 0$) as follows,

$$D = A_d \cdot \left[(\boldsymbol{\alpha}^d - \boldsymbol{\alpha}) : \mathbf{n} \right] (1 + \langle \mathbf{z} : \mathbf{n} \rangle) \quad (85)$$

This relationship enhances the volumetric contraction whenever the fabric is favorable ($\mathbf{z} : \mathbf{n} \geq 0$), based on the term $1 + \langle \mathbf{z} : \mathbf{n} \rangle$ as recommended by Dafalias and Manzari (2004).

The effect of fabric on dilatancy during contraction was modified for the present model as,

$$D = A_{dc} \cdot \left[(\boldsymbol{\alpha} - \boldsymbol{\alpha}_{in}^{app}) : \mathbf{n} + C_{in} \right]^2 \frac{(\boldsymbol{\alpha}^d - \boldsymbol{\alpha}) : \mathbf{n}}{(\boldsymbol{\alpha}^d - \boldsymbol{\alpha}) : \mathbf{n} + C_D} C_{p\min 2} \quad (86)$$

$$A_{dc} = \frac{A_{do} (1 + \langle \mathbf{z} : \mathbf{n} \rangle)}{h_p C_{dz}} \quad (87)$$

$$C_D = 0.1 \quad (88)$$

$$C_{in} = \frac{\sqrt{2} \langle \mathbf{z} : \mathbf{n} \rangle}{z_{\max}} \quad (89)$$

$$C_{dz} = \left(1 - C_{rot2} \cdot \frac{z_{peak}}{z_{\max}} \right) \cdot \left(\frac{z_{\max}}{z_{\max} + C_{rot2} \cdot z_{cum}} \right) \geq \frac{1}{1 + \frac{z_{\max}}{2}} \quad (90)$$

$$C_{rot2} = 1 - \frac{z_{peak}}{z_{cum} + \frac{z_{\max}}{100}} (= 1 - C_{zpk2}) \quad (91)$$

$$\begin{aligned} C_{p\min 2} &= 0 \quad \text{for } p \leq 2p_{\min} \\ C_{p\min 2} &= 1 \quad \text{for } p \geq 18p_{\min} \\ C_{p\min 2} &= \frac{p - 2p_{\min}}{16p_{\min}} \quad \text{otherwise} \end{aligned} \quad (92)$$

The factor C_{in} in the expression for D has been modified so it now depends on fabric; C_{in} is zero for unfavorable fabric and increases with increasing $z:n$ for favorable fabric to enhance the contraction rate at the start of an unloading cycle (note that D would be zero at the start of an unloading cycle if C_{in} was zero).

The term C_{dz} in the denominator of the expression for A_{dc} serves to increase the rate of contraction as z_{peak} nears z_{max} or as a large amount of cumulative fabric formation/destruction has taken place. This term was developed for improved modeling of the cyclic strength of denser sands, for which the value of h_p can be on the order of 100 (Figure 2.13). The degrading of the denominator as z_{peak} or z_{cum} increases enables the generation of high excess pore pressures at higher loading levels and controls the slope of the CRR versus number of uniform loading cycles relationship obtained for undrained element loading. Note that the denominator degrades whether fabric is favorable or not, but that the overall rate of contraction is more enhanced if the fabric is favorable ($z:n \geq 0$). The factor C_{rot2} was introduced into the factor C_{dz} to provide better control over the rate of contraction as z_{peak} nears z_{max} or as a large amount of cumulative fabric formation/destruction has taken place. The factor C_{rot2} takes values that range from 1 for loading with zero fabric or cyclic loading that causes reversals of fabric (since z_{cum} will become much larger than z_{peak}), to 0 for loading that causes fabric to grow monotonically in one direction such as in non-reversal cyclic loading (since z_{cum} will equal z_{peak}). Lastly, the limit on the minimum value of C_{dz} is required for avoiding division by zero and to avoid overestimating contraction rates (i.e., small values of h_p and large values of z_{peak} or z_{cum}).

The term C_{pmin2} slows the rate of contraction when p is approaching its minimum allowable value, and stops further contraction when p is less than twice the minimum allowable p .

Effect of fabric on the elastic modulus

The elastic shear modulus and elastic bulk modulus may degrade with increasing values of cumulative plastic deviator strain term, z_{cum} . This component of the model was added to account for the progressive destruction, with increasing plastic shear strains, of any minor cementation bonds or other ageing- or strain history-related phenomena that produced an increase in small-strain shear modulus. The destruction of minor cementation by plastic shear strains is evidenced in the field by measurements of shear wave velocities in sand that are lower after earthquake shaking than before earthquake shaking (e.g., Arai 2006). The degradation of the elastic shear modulus is computed as,

$$G = G_o p_A \left(\frac{p}{p_A} \right)^{\frac{1}{2}} C_{SR} \left(\frac{1 + \frac{z_{cum}}{z_{max}}}{1 + \frac{z_{cum}}{z_{max}} C_{GD}} \right) \quad (93)$$

where C_{GD} is the factor by which the shear modulus is degraded (divided) at very large values of z_{cum} . This change in the elastic shear modulus G causes the bulk modulus K to also progressively decrease with increasing z_{cum} . The change in K improves the model's ability to track the stress-strain response of liquefying sand. In particular, decreasing K with increasing z_{cum} reduces the rate of strain-hardening after phase transformation at larger shear strain levels, and improves the ability to approximate the hysteretic stress-strain response of a soil as it liquefies.

2.9 Post-shaking reconsolidation

Volumetric strains that develop during post-liquefaction reconsolidation of sand are challenging to numerically model using the conventional constitutive separation of strains into elastic and plastic components since a large portion of the post-liquefaction reconsolidation strains are due to sedimentation effects which are not easily incorporated into either the elastic or plastic components of behavior. This limitation is further compounded in stress ratio-controlled models with open yield surfaces, such as PM4Sand, since the constant stress ratio path of reconsolidation stays within the elastic region. Single element simulations using various constitutive models show that they generally predict post-liquefaction reconsolidation strains that are an order of magnitude smaller than observed in various experimental studies (e.g., Ziotopoulou and Boulanger 2013b, Howell et al. 2014).

The present model was modified to provide more realistic estimates of reconsolidation strains during the post-shaking portion of a numerical simulation. The modification involved the pragmatic approach of reducing the post-shaking elastic shear modulus G (and hence elastic bulk modulus K) which increases reconsolidation strains, thereby compensating for the sedimentation strains which are not explicitly modeled. The user may activate this feature after the end of strong shaking, such that post-liquefaction reconsolidation strains are better approximated in the remainder of the simulation. This feature should not be activated for the strong shaking portion of a simulation.

The post-shaking elastic moduli are determined by multiplying the conventional elastic moduli (computed using the expressions described earlier) by a reduction factor F_{sed} as,

$$G_{post-shaking} = F_{sed} G \quad (94)$$

$$K_{post-shaking} = F_{sed} K \quad (95)$$

The F_{sed} value is computed as,

$$F_{sed} = F_{sed,min} + (1 - F_{sed,min}) \left(\frac{p'}{20p'_{sed}} \right)^2 \leq 1 \quad (96)$$

$$p'_{sed} = p'_{sed,o} \left(\frac{z_{cum}}{z_{cum} + z_{max}} \right) \left\langle 1 - \frac{M^{cur}}{M^d} \right\rangle^{0.25} \quad (97)$$

$$F_{sed,min} = 0.04 + 0.0625(D_R - 0.35) \leq 0.99 \quad (98)$$

$$p'_{sed,o} = -\frac{P_{atm}}{5} \quad (99)$$

where the constant value $F_{sed,min}$ represents the smallest value that F_{sed} can attain and is dependent on the initial relative density, and the parameter $p'_{sed,o}$ is the mean effective stress up to which reconsolidation strains are enhanced. The value of F_{sed} progressively reduces from unity toward the

value of $F_{sed,min}$ as z_{cum} progressively increases and provided that M^{cur} is less than M^d . Setting $F_{sed,min} = 0.04$ was found to produce reasonable responses as shown later. The user can select values for $p'_{sed,o}$ and $F_{sed,min}$.

2.10 Summary of constitutive equations

The constitutive equations for the model presented herein are summarized in Table 2.1 along with the equations for the Dafalias-Manzari (2004) model.

Table 2.1. Comparison of constitutive equations

Dafalias-Manzari (2004) model	Present model
Critical state line $e_c = e_o - \lambda \left(\frac{p_{cs}}{p_A} \right)^\xi$	Critical state line $\xi_R = \frac{R}{Q - \ln \left(100 \frac{p}{p_A} \right)} - D_R$
Elastic deviatoric strain increment $de^{el} = \frac{ds}{2G}$ $G = G_o p_A \frac{(2.97 - e)^2}{1 + e} \left(\frac{p}{p_A} \right)^{1/2}$	Elastic deviatoric strain increment $de^{el} = \frac{ds}{2G}$ $G = G_o p_A \left(\frac{p}{p_A} \right)^{1/2} C_{SR} \left(\frac{1 + \frac{z_{cum}}{z_{max}}}{1 + \frac{z_{cum}}{z_{max}} C_{GD}} \right)$ $C_{SR} = \frac{1 - C_{SR,o} \cdot \left(\frac{M}{M^b} \right)^{m_{SR}}}{\left 1 - C_{SR,o} \cdot \left(\frac{M}{M^b} \right)^{m_{SR}} \right _{initial}} \leq 1.0$ $C_{SR,o} = 0.5$ $m_{SR} = 4$
Elastic volumetric strain increment $d\varepsilon_v^{el} = \frac{dp}{K}$	Elastic volumetric strain increment $d\varepsilon_v^{el} = \frac{dp}{K}$

$K = \frac{2(1+\nu)}{3(1-2\nu)} G$	$K = \frac{2(1+\nu)}{3(1-2\nu)} G$
<i>Yield surface</i> $f = [(s - p\alpha) : (s - p\alpha)]^{1/2} - \sqrt{2/3} pm = 0$	<i>Yield surface</i> $f = [(s - p\alpha) : (s - p\alpha)]^{1/2} - \sqrt{2/3} pm = 0$

Plastic deviatoric strain increment

$$d\mathbf{e}^{pl} = \langle L \rangle \mathbf{R}'$$

$$\mathbf{R} = B\mathbf{n} - C \left(\mathbf{n}^2 - \frac{1}{3} \mathbf{I} \right) + \frac{1}{3} D \mathbf{I}$$

$$B = 1 + \frac{3}{2} \frac{1-c}{c} g(\theta, c) \cos(3\theta)$$

$$C = 3 \sqrt{\frac{3}{2}} \frac{1-c}{c} g(\theta, c)$$

$$c = \frac{Q_{ext}}{Q_{compr}}$$

$$g(\theta, c) = \frac{2c}{(1+c) - (1-c)\cos(3\theta)}$$

$$M^b = M \cdot \exp(-n^b \xi_R)$$

$$\alpha_\theta^b = \sqrt{\frac{2}{3}} [g(\theta, c) M^b - m] \mathbf{n}$$

$$K_p = \frac{2}{3} G \cdot h_o \cdot \frac{(\alpha_\theta^b - \alpha) : \mathbf{n}}{(\alpha - \alpha_{in}) : \mathbf{n}}$$

Plastic deviatoric strain increment

$$d\mathbf{e}^{pl} = \langle L \rangle \mathbf{R}'$$

$$\mathbf{R} = B\mathbf{n} - C \left(\mathbf{n}^2 - \frac{1}{3} \mathbf{I} \right) + \frac{1}{3} D \mathbf{I}$$

$$B = 1 + \frac{3}{8} \frac{c^4 - 1}{c^4} g(\theta, c)^4 \cos(3\theta)$$

$$C = \frac{3\sqrt{6}}{8} \frac{c^4 - 1}{c^4} g(\theta, c)^4$$

$$g(\theta, c) = \frac{2^{n_s} c}{\left[\left(1 + c^{1/n_s} \right) - \left(1 - c^{1/n_s} \right) \cos(3\theta) \right]^{n_s}}$$

$$n_s = 0.25$$

For dense-of-critical states:

$$M^b = M \cdot g \cdot \exp(-n^b \xi_R)$$

For loose-of-critical states:

$$M^b = M \cdot g \cdot \exp\left(-\frac{n^b}{4} \xi_R\right)$$

$$M = \frac{6 \cdot \sin(\phi_{cv})}{3 - \sin(\phi_{cv})}$$

$$\alpha^b = \sqrt{\frac{2}{3}} [M^b - m] \mathbf{n}$$

	$K_p = G \cdot h_o \cdot g^{-n_g} \frac{\left[(\boldsymbol{a}^b - \boldsymbol{a}) : \boldsymbol{n} \right]^{0.5}}{\left[\exp((\boldsymbol{a} - \boldsymbol{a}_{in}) : \boldsymbol{n}) - 1 \right] + C_{\gamma 1}} C_{rev}$ $\cdot \frac{C_{K\alpha}}{1 + C_{Kp} \left(\frac{z_{peak}}{z_{max}} \right) \langle (\boldsymbol{a}^b - \boldsymbol{a}) : \boldsymbol{n} \rangle \sqrt{1 - C_{zpk2}}}$ $C_{rev} = \frac{(\boldsymbol{a} - \boldsymbol{a}_{in}^{app}) : \boldsymbol{n}}{(\boldsymbol{a} - \boldsymbol{a}_{in}^{true}) : \boldsymbol{n}} \quad for \quad (\boldsymbol{a} - \boldsymbol{a}_{in}^p) : \boldsymbol{n} \leq 0$ $= 1 \quad otherwise$ $C_{K\alpha} = 1 + \frac{C_{K\alpha f}}{1 + \left(2.5 \cdot \langle (\boldsymbol{a} - \boldsymbol{a}_{in}^{true}) : \boldsymbol{n} \rangle \right)^2} \cdot C_{pzp2} \cdot C_{zpk1}$ $C_{zpk1} = \frac{z_{peak}}{z_{cum} + \frac{z_{max}}{5}}$ $C_{zpk2} = \frac{z_{peak}}{z_{cum} + \frac{z_{max}}{100}}$ $C_{pzp2} = \frac{-\langle -(p_{zp} - p) \rangle}{-\langle -(p_{zp} - p) \rangle + p_{min}}$ $C_{\gamma 1} = \frac{h_o}{200}$ $C_{Kp} = \sqrt{2}$
Plastic volumetric strain increment $d\varepsilon_v^{pl} = \langle L \rangle D$	Plastic volumetric strain increment $d\varepsilon_v^{pl} = \langle L \rangle D$

$$M^d = M \cdot \exp(n^d \xi_R)$$

$$\alpha_\theta^d = \sqrt{2/3} [g(\theta, c) M^d - m] \mathbf{n}$$

$$D = A_d \cdot [(\alpha_\theta^d - \alpha) : \mathbf{n}]$$

$$A_d = A_o (1 + \langle \mathbf{z} : \mathbf{n} \rangle)$$

For dense-of-critical states:

$$M^d = M \cdot g \cdot \exp(n^d \xi_R)$$

For loose-of-critical states:

$$M^d = M \cdot g \cdot \exp(4n^d \xi_R)$$

$$M^{dR} = \frac{M^d}{C_{rot1}}$$

$$C_{rot1} = 1 + \frac{\sqrt{2} \cdot \langle -\mathbf{z} : \mathbf{n} \rangle}{z_{\max}} \cdot (1 - C_{zin1}) \geq 1$$

$$C_{zin1} = \left\langle 1 - \exp \left(-2.0 \left| \frac{\mathbf{z}_{in} : \mathbf{n} - \mathbf{z} : \mathbf{n}}{z_{\max}} \right| \right) \right\rangle$$

$$\alpha^d = \sqrt{\frac{2}{3}} \cdot (M^d - m) \mathbf{n}$$

$$\alpha^{dR} = \sqrt{\frac{2}{3}} \cdot (M^{dR} - m) \mathbf{n}$$

If dilating ($D < 0$):

$$D_{non-rot} = A_d \cdot [(\alpha^d - \alpha) : \mathbf{n}]$$

$$D_{rot} = A_d \cdot \frac{\langle -\mathbf{z} : \mathbf{n} \rangle}{z_{\max}} \cdot \frac{(\alpha^{dR} - \alpha) : \mathbf{n}}{C_{DR}}$$

$$\text{if } D_{non-rot} < D_{rot} \Rightarrow D = D_{non-rot}$$

$$\text{else } D = D_{non-rot} + (D_{rot} - D_{non-rot}) \cdot \frac{\langle M^b - M^{cur} \rangle}{\langle M^b - M^{cur} + 0.01 \rangle}$$

$$A_d = \frac{A_{do} \cdot g^{-n_g} \cdot C_{zin2}}{\left(\frac{z_{cum}^2}{2z_{max}}\right) \left(1 - \frac{\langle -z : n \rangle}{z_{peak}}\right)^3 (C_\varepsilon)^2 (C_{pzp})(C_{pmin})(C_{zin1}) + 1}$$

$$A_{do} = 3.125 \frac{\left[\sin^{-1}\left(\frac{3M^b}{6+M^b}\right) - \sin^{-1}\left(\frac{3M}{6+M}\right)\right]}{(M^b - M^d)}$$

$$C_{pzp} = \frac{1}{1 + \left(2.5p / p_{zp}\right)^5}$$

$$C_{pmin} = \frac{1}{1 + \left(p_{min} / p\right)^2}$$

$$C_{zin1} = \left\langle 1 - \exp\left(-2.0 \left| \frac{z_{in} : n - z : n}{z_{max}} \right| \right) \right\rangle$$

$$C_{zin2} = \frac{1 + C_{zin1} \frac{z_{cum} - z_{peak}}{3z_{max}}}{1 + 3C_{zin1} \frac{z_{cum} - z_{peak}}{3z_{max}}}$$

If contracting ($D \geq 0$)

$$D = A_{dc} \cdot \left[(\alpha - \alpha_{in}^{app}) : n + C_{in} \right]^2 \frac{(\alpha^d - \alpha) : n}{(\alpha^d - \alpha) : n + C_D} C_{pmin2} \leq 1.5 \cdot A_{do} \frac{(\alpha^d - \alpha) : n}{(\alpha^d - \alpha) : n + C_D}$$

$$A_{dc} = \frac{A_{do} (g^{-n_g}) (1 + \langle z : n \rangle)}{h_p C_{dz}}$$

	$C_{in} = \frac{\sqrt{2} \langle \mathbf{z} : \mathbf{n} \rangle}{z_{max}}$ $C_{dz} = \left(1 - C_{rot2} \cdot \frac{\sqrt{2} z_{peak}}{z_{max}} \right) \cdot \left(\frac{z_{max}}{z_{max} + C_{rot2} z_{cum}} \right) \geq \frac{1}{1 + \frac{z_{max}}{2}}$ $C_{rot2} = 1 - \frac{z_{peak}}{z_{cum} + \frac{z_{max}}{100}} \left(= 1 - C_{zpk2} \right)$ $C_D = 0.1$ $C_{pmin2} = 0 \quad \text{for } p \leq 2p_{min}$ $C_{pmin2} = 1 \quad \text{for } p \geq 18p_{min}$ $C_{pmin2} = \frac{p - 2p_{min}}{16p_{min}} \quad \text{otherwise}$ $h_p = h_{po} \exp \left(0.9 + 6.1 (0.5 - \xi_R)^{2.1} \right) \quad \text{for } \xi_R \leq 0.5$ $h_p = h_{po} \exp(0.9) \quad \text{for } \xi_R > 0.5$
--	---

<i>Fabric-dilatancy tensor update</i> $d\mathbf{z} = -c_z \langle -d\varepsilon_v^{pl} \rangle (z_{max} \mathbf{n} + \mathbf{z})$	<i>Fabric-dilatancy tensor update if $(\alpha^d - \alpha) : \mathbf{n} < 0$</i> $d\mathbf{z} = -\frac{c_z}{1 + \left\langle \frac{z_{cum}}{2\sqrt{2}z_{max}} - 1 \right\rangle} \frac{\langle -d\varepsilon_v^{pl} \rangle}{D} (z_{max} \mathbf{n} + \mathbf{z})$ $dz_{cum} = d\mathbf{z} $ $z_{peak} = \max(\sqrt{\mathbf{z} : \mathbf{z}}, z_{peak})$
---	--

Stress increment $L = \frac{2G\mathbf{n} : d\mathbf{e} - \mathbf{n} : \mathbf{r}Kd\varepsilon_v}{K_p + 2G(B - C \cdot tr\mathbf{n}^3) - KD\mathbf{n} : \mathbf{r}}$ $d\boldsymbol{\sigma} = 2Gd\mathbf{e} + Kd\varepsilon_v\mathbf{I}$ $-\langle L \rangle \left(2G \left[B\mathbf{n} \left(\mathbf{n}^2 - \frac{1}{3} \right) \mathbf{I} \right] + KDI \right)$	Stress increment $L = \frac{2G\mathbf{n} : d\mathbf{e} - \mathbf{n} : \mathbf{r}Kd\varepsilon_v}{K_p + 2G(B - C \cdot tr\mathbf{n}^3) - KD\mathbf{n} : \mathbf{r}}$ $d\boldsymbol{\sigma} = 2Gd\mathbf{e} + Kd\varepsilon_v\mathbf{I} - \langle L \rangle \left(2G \left[B\mathbf{n} \left(\mathbf{n}^2 - \frac{1}{3} \right) \mathbf{I} \right] + KDI \right)$
Back stress increment $d\boldsymbol{\alpha} = \frac{2}{3} \langle L \rangle h(\boldsymbol{\alpha}^b - \boldsymbol{\alpha}_0)$	Back stress increment $d\boldsymbol{\alpha} = \frac{\langle L \rangle K_p}{p_0} \cdot \frac{\boldsymbol{\alpha}^b - \boldsymbol{\alpha}_0}{(\boldsymbol{\alpha}^b - \boldsymbol{\alpha}_0 : \mathbf{n})}$
	Post-shaking reconsolidation $G_{post-shaking} = F_{sed} G$ $K_{post-shaking} = F_{sed} K$ $F_{sed} = F_{sed,min} + (1 - F_{sed,min}) \left(\frac{p'}{20 p'_{sed}} \right) \leq 1$ $p'_{sed} = p'_{sed_o} \left(\frac{z_{cum}}{z_{cum} + z_{max}} \right) \left\langle 1 - \frac{M^{cur}}{M^d} \right\rangle^{0.25} \quad p'_{sed_o} = -\frac{P_{atm}}{5}$ $F_{sed,min} = 0.04 + 0.0625(D_R - 0.35) \leq 0.99$

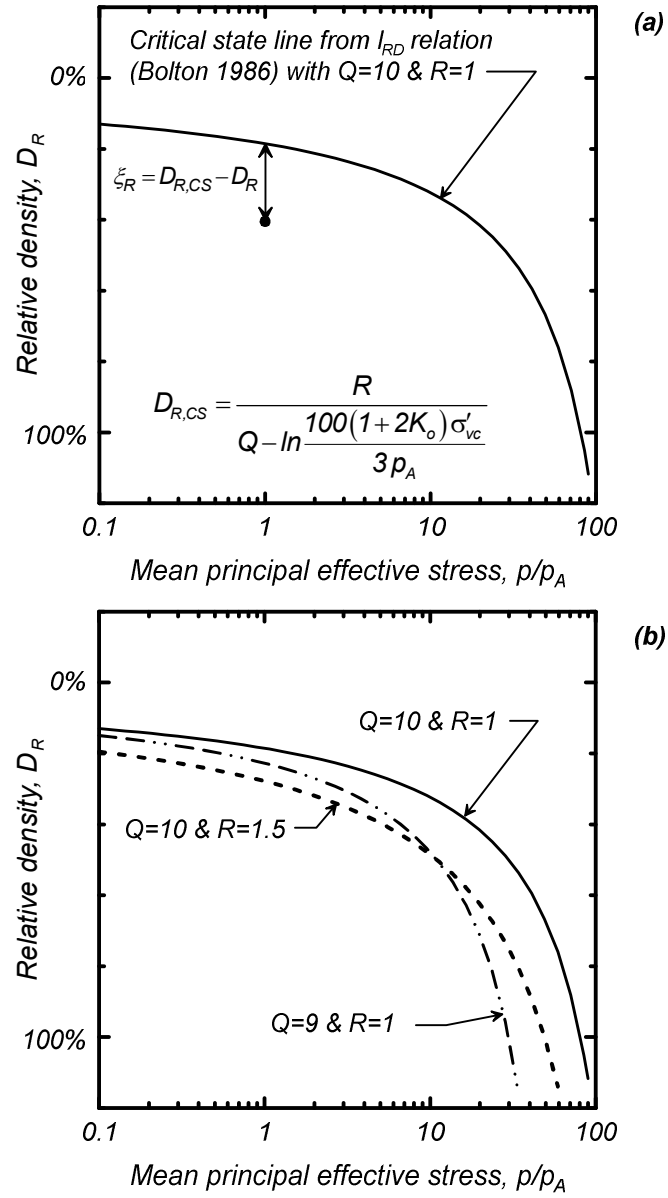


Figure 2.1. Definition of the relative state parameter index, ξ_R (Boulanger 2003a) and the effects of varying Q and R .

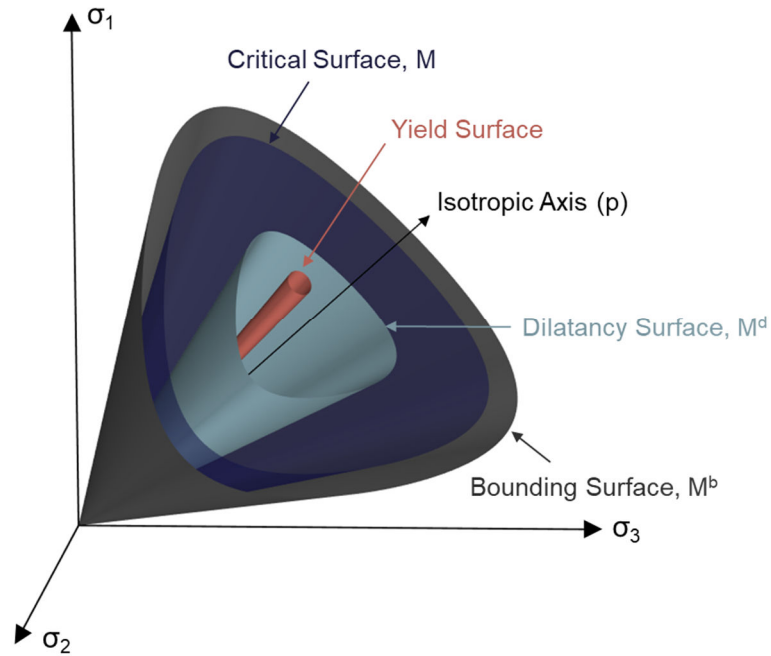


Figure 2.2. Schematic of yield, critical, dilatancy, and bounding surfaces in principal stress space. Relative location of dilatancy and bounding surfaces corresponds to dense-of-critical states of stress.

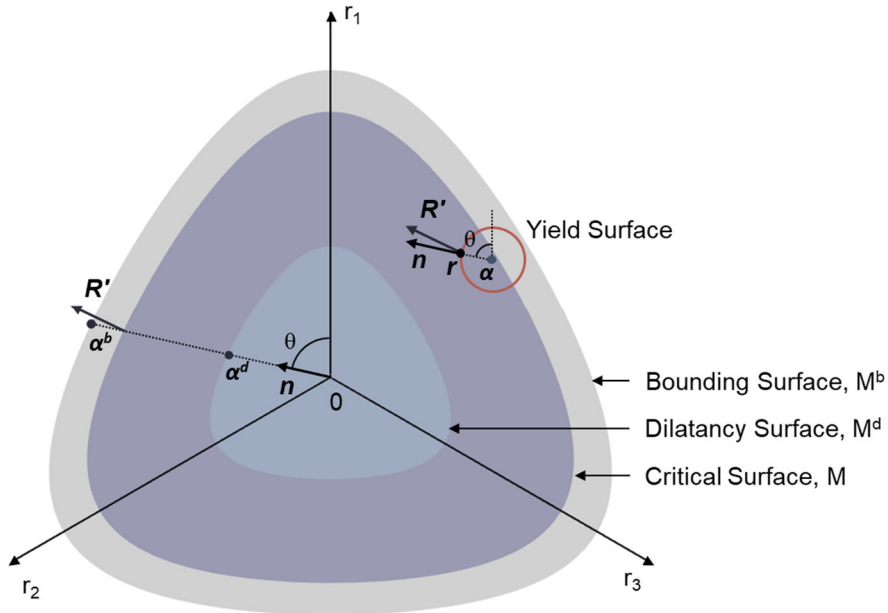


Figure 2.3. Schematic of the bounding, dilatancy, and critical surfaces on the stress ratio π plane with the yield surface, normal tensor, dilatancy back stress ratio, and bounding back stress ratio (after Dafalias and Manzari 2004). Relative location of the yield surface differs from that in Figure 2.2.

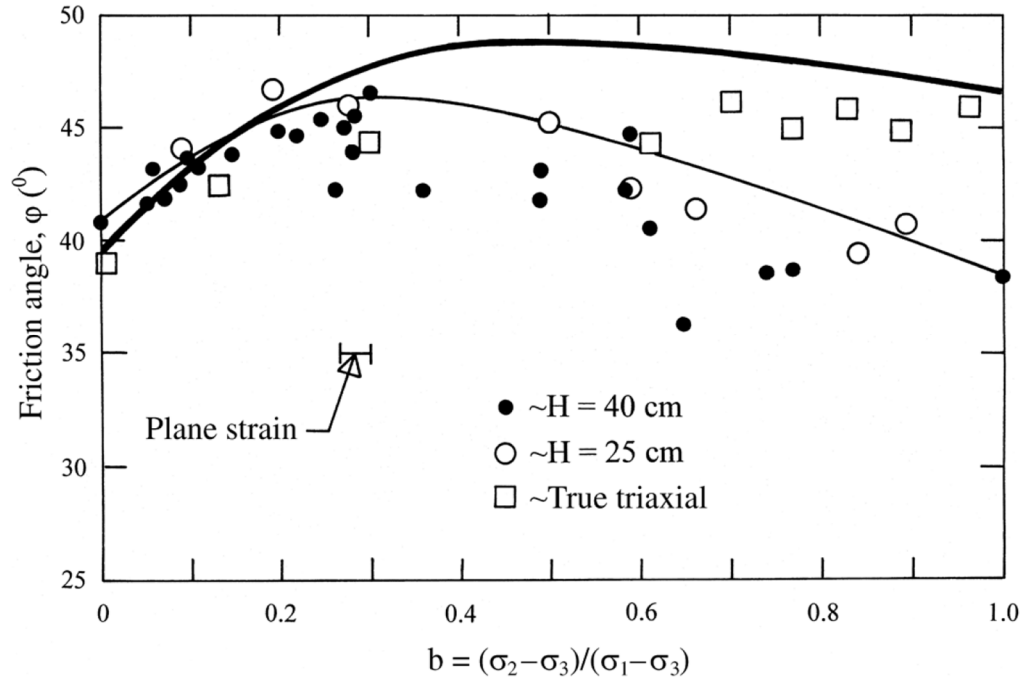


Figure 2.4. Friction angles versus intermediate principal stress ratio from torsional shear (specimen heights of 25 and 40 cm) and true triaxial tests on Santa Monica beach sand. The thick solid line is an isotropic failure surface model, whereas the thin solid line is a trend line for the upper range of friction angles from the torsion shear tests (Lade et al. 2008).

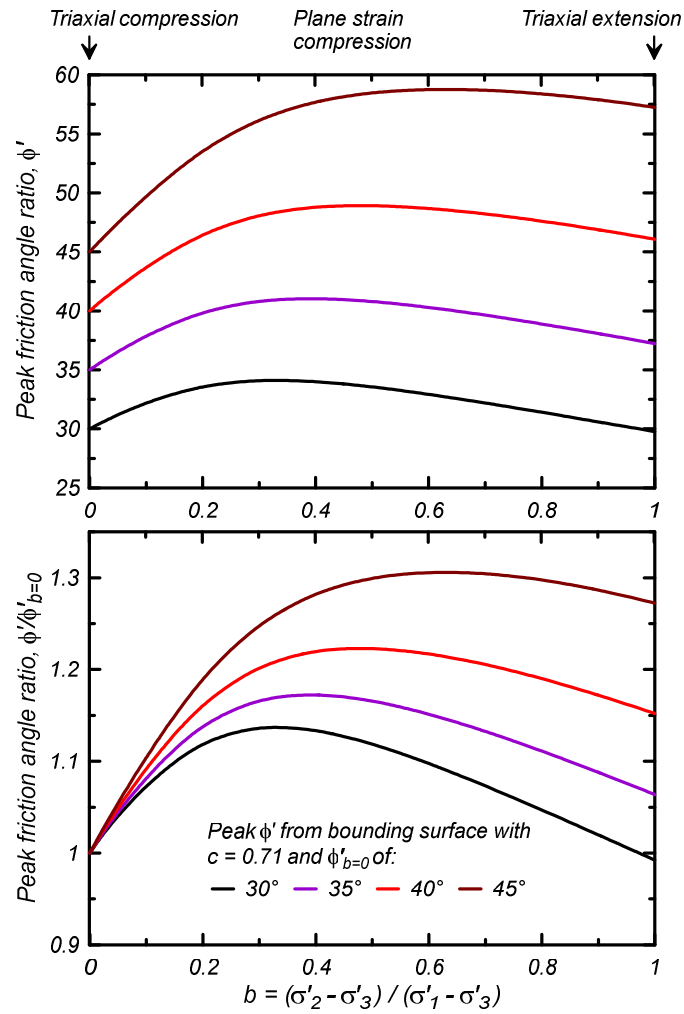


Figure 2.7. Variation of peak friction angle with intermediate principal stress ratio with $c = 0.71$ and a range of peak friction angles in triaxial compression.

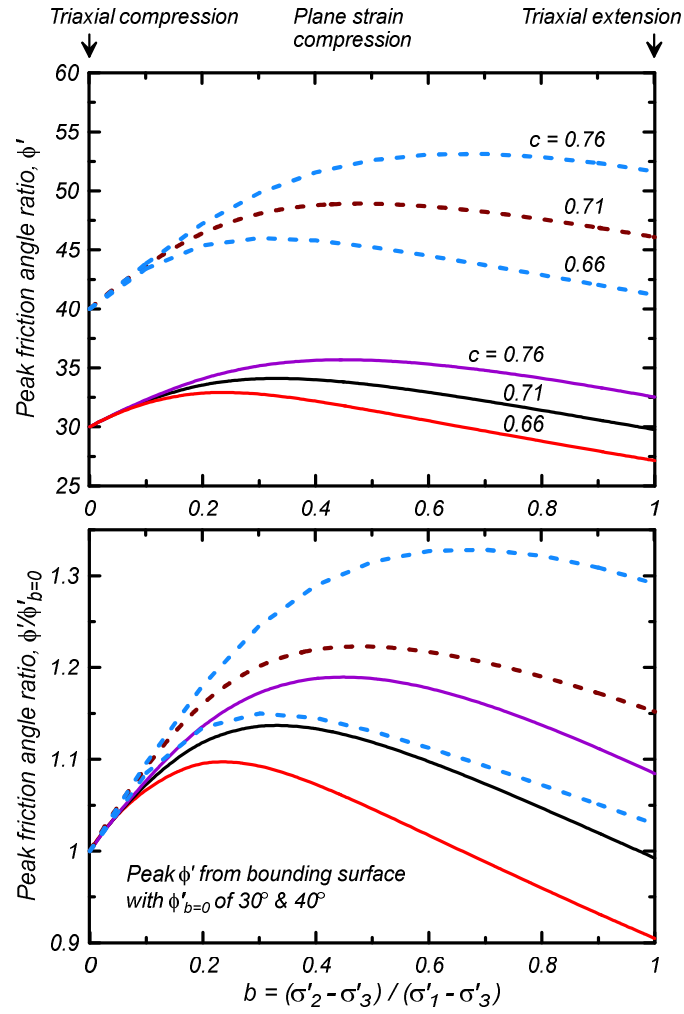


Figure 2.8. Variation of peak friction angle with intermediate principal stress ratio for $c = 0.66, 0.71, 0.76$ with peak friction angles in triaxial compression of 30° and 40° .

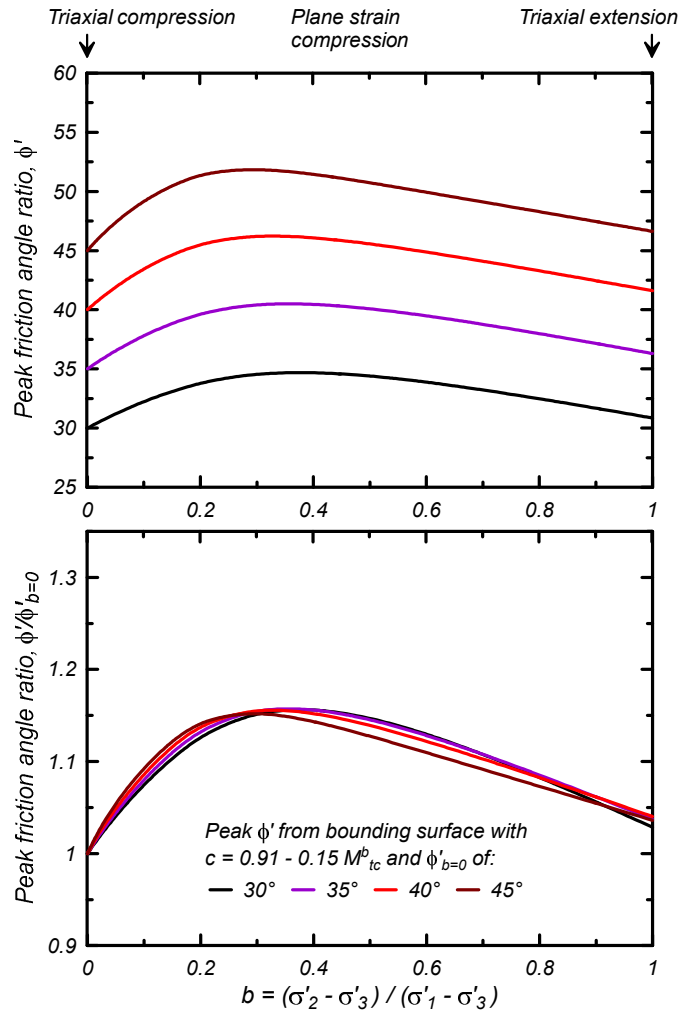


Figure 2.9. Variation of peak friction angle with intermediate principal stress ratio using the default function for c and a range of peak friction angles in triaxial compression.

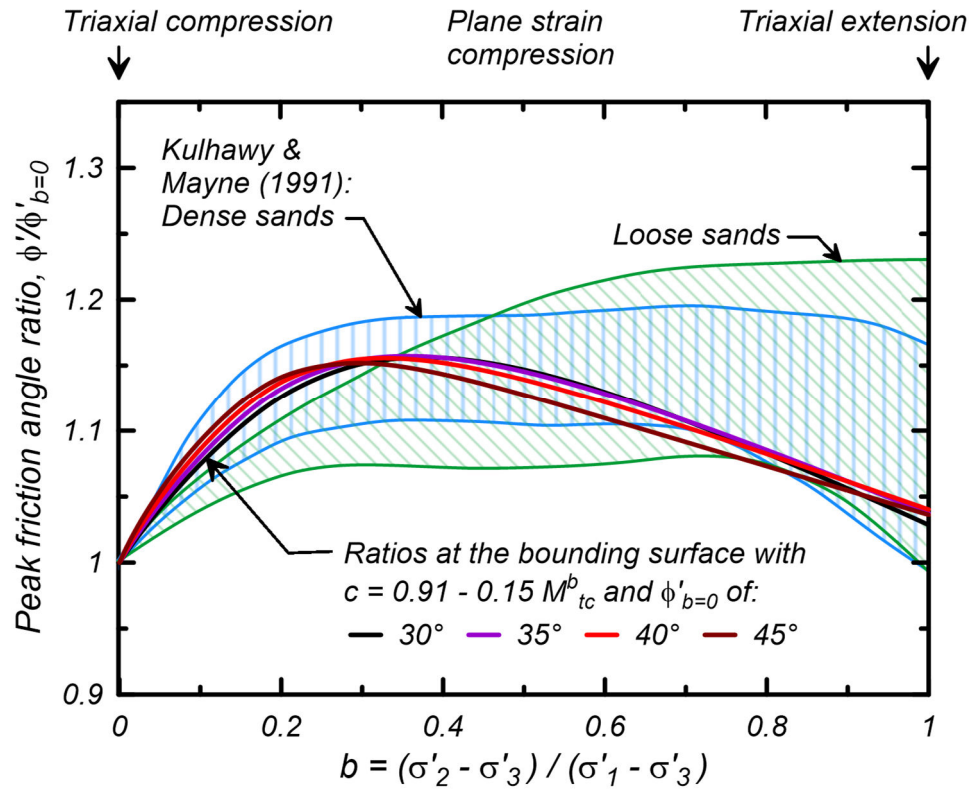


Figure 2.10. Variation of friction angle ratio with intermediate principal stress ratio: present model with default function for c and the experimental ranges for dense and loose sands from Kulhawy and Mayne (1990).

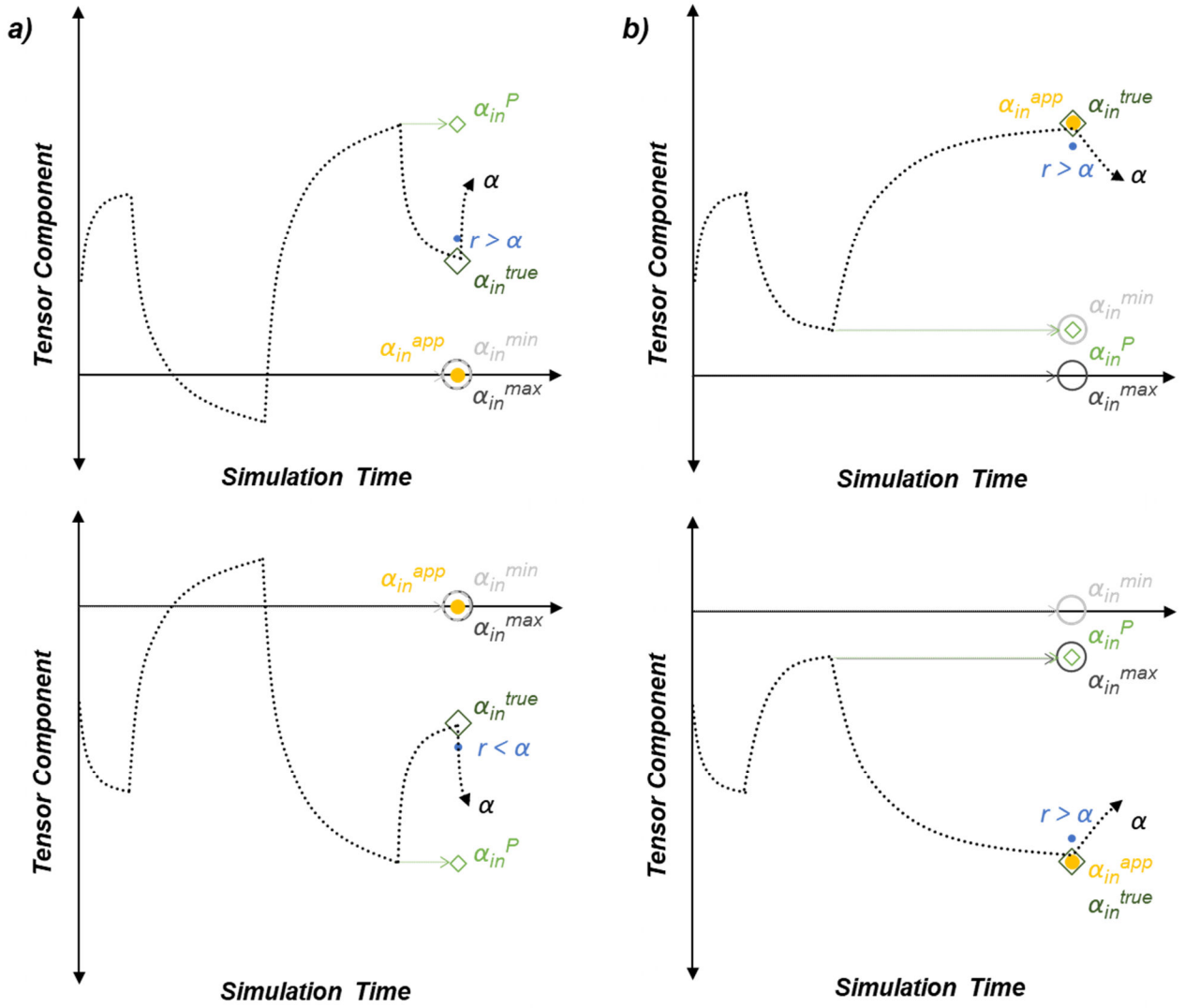


Figure 2.11. Schematic showing definitions of back-stress ratio tensors using a single component over the course of a simulation for: (a) two loading histories that begin with reversals in the sign of the back-stress component, and (b) two loading histories with recent loading reversals that do not involve reversal of the sign of the back-stress component.

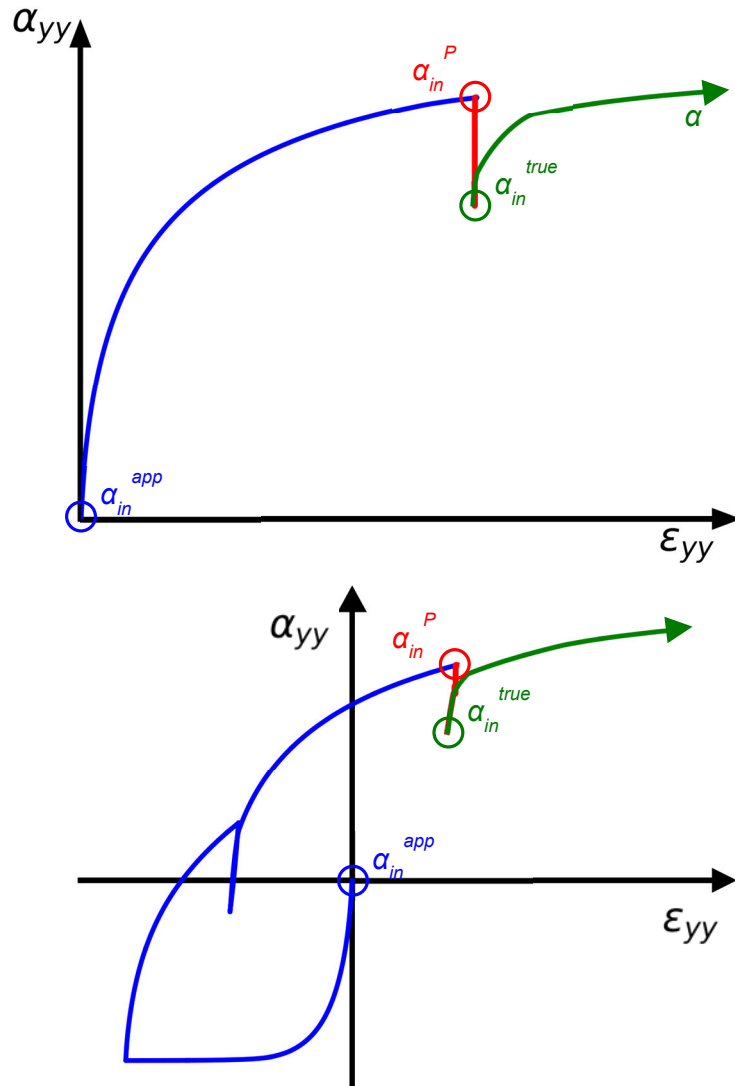


Figure 2.12. Drained triaxial simulations showing α_{yy} versus ϵ_{yy} with the points corresponding to the current back-stress ratio α , the apparent initial back-stress ratio α_{in}^{app} , the true initial back-stress ratio α_{in}^{true} , and the previous initial back-stress ratio α_{in}^P for: (a) monotonic shearing with one intermediate unload-reload cycle, and (b) a more general sequence of cyclic loading.

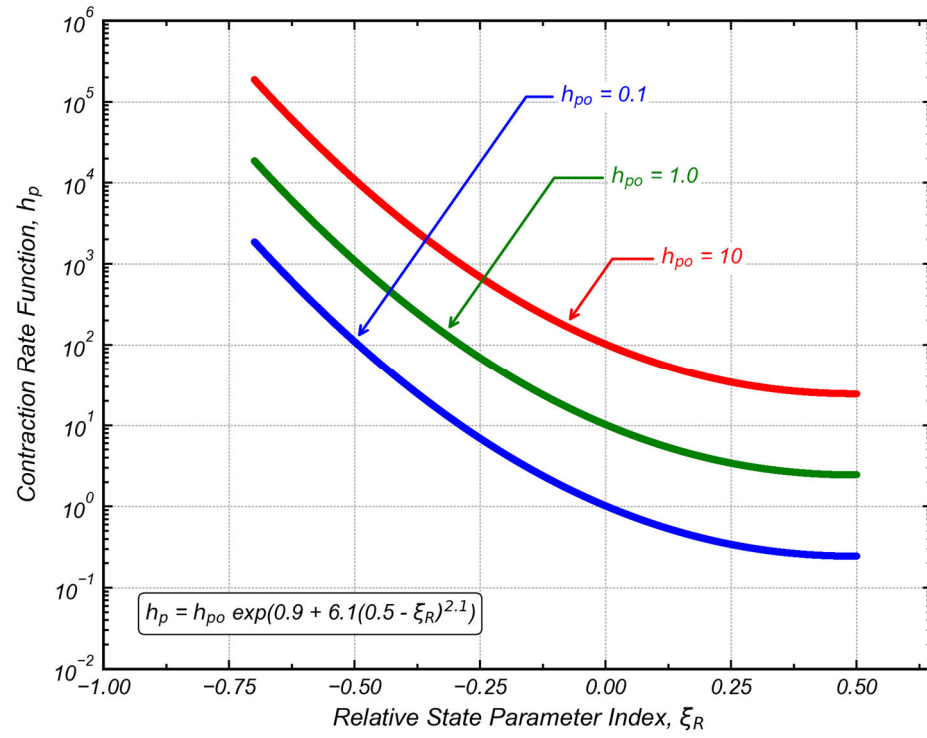


Figure 2.13. Variation of contraction rate function h_p with relative state parameter index D_R and the contraction rate parameter h_{po} .

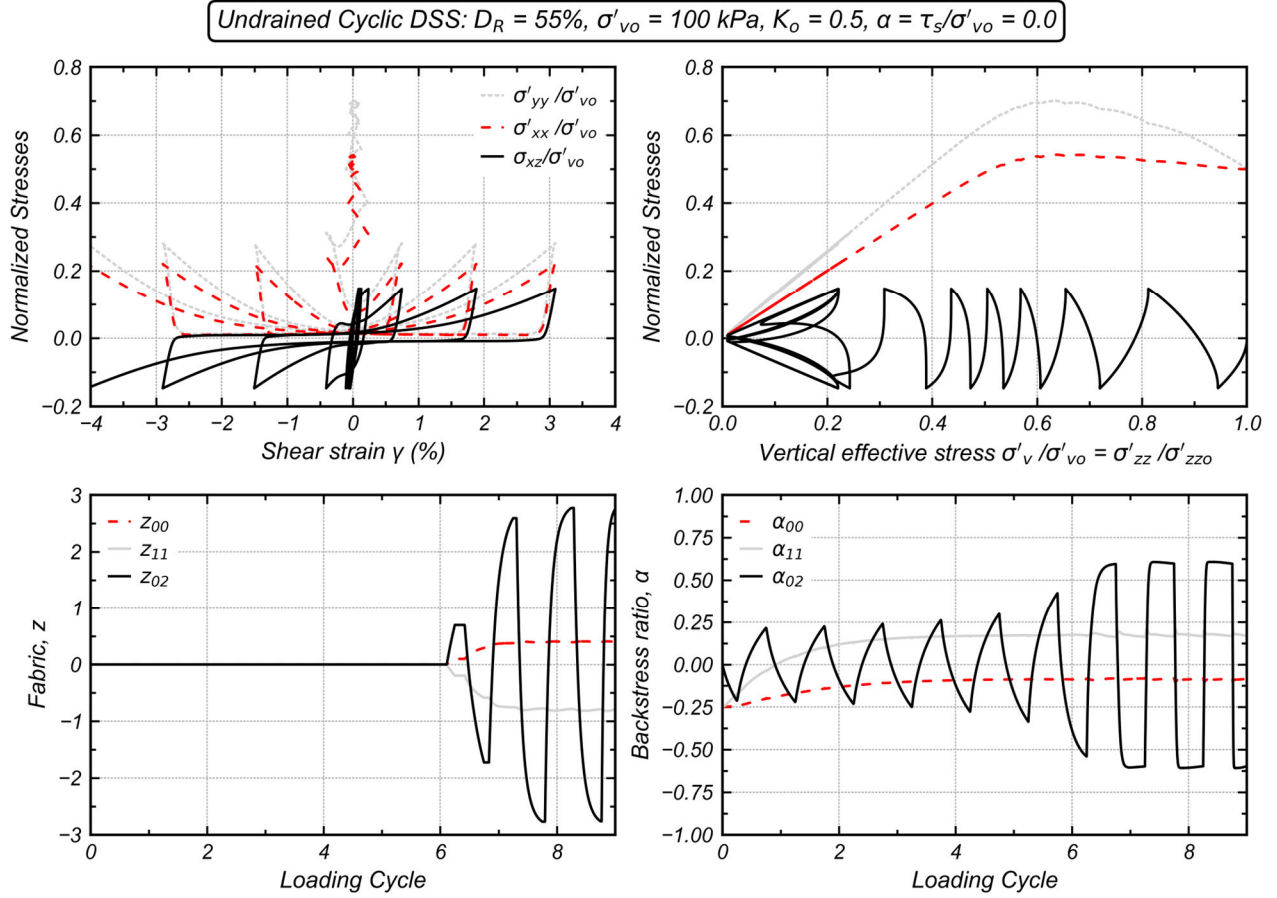


Figure 2.14. Example undrained cyclic DSS loading response for $D_R = 55\%$ with an initial static shear stress ratio of $\alpha=0.0$, showing the variation in stresses, back-stress ratios, and fabric tensor terms. Note that z corresponds to the vertical direction.

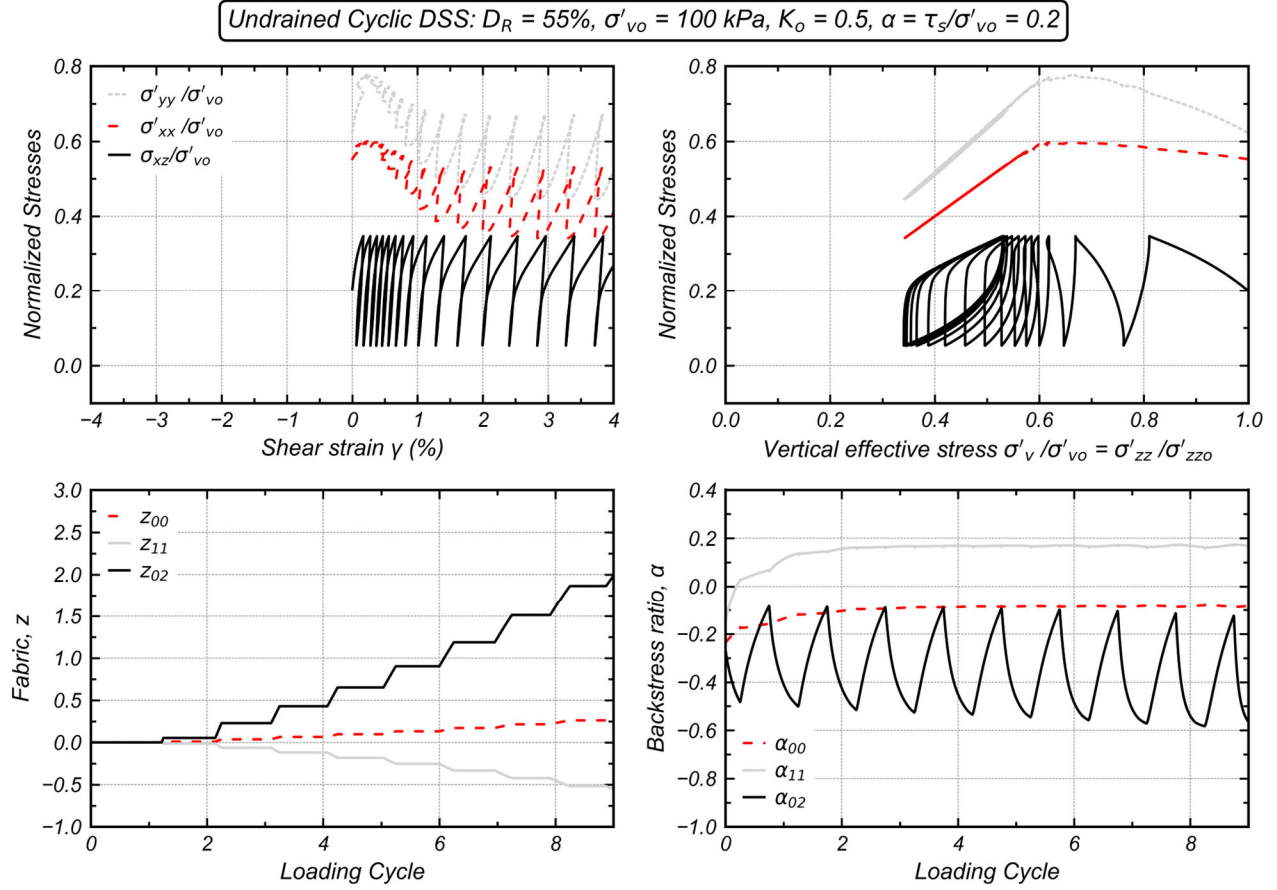


Figure 2.15. Example undrained cyclic DSS loading response for $D_R = 55\%$ with an initial static shear stress ratio of $\alpha=0.20$, showing the variation in stresses, back-stress ratios, and fabric tensor terms. Note that z corresponds to the vertical direction.

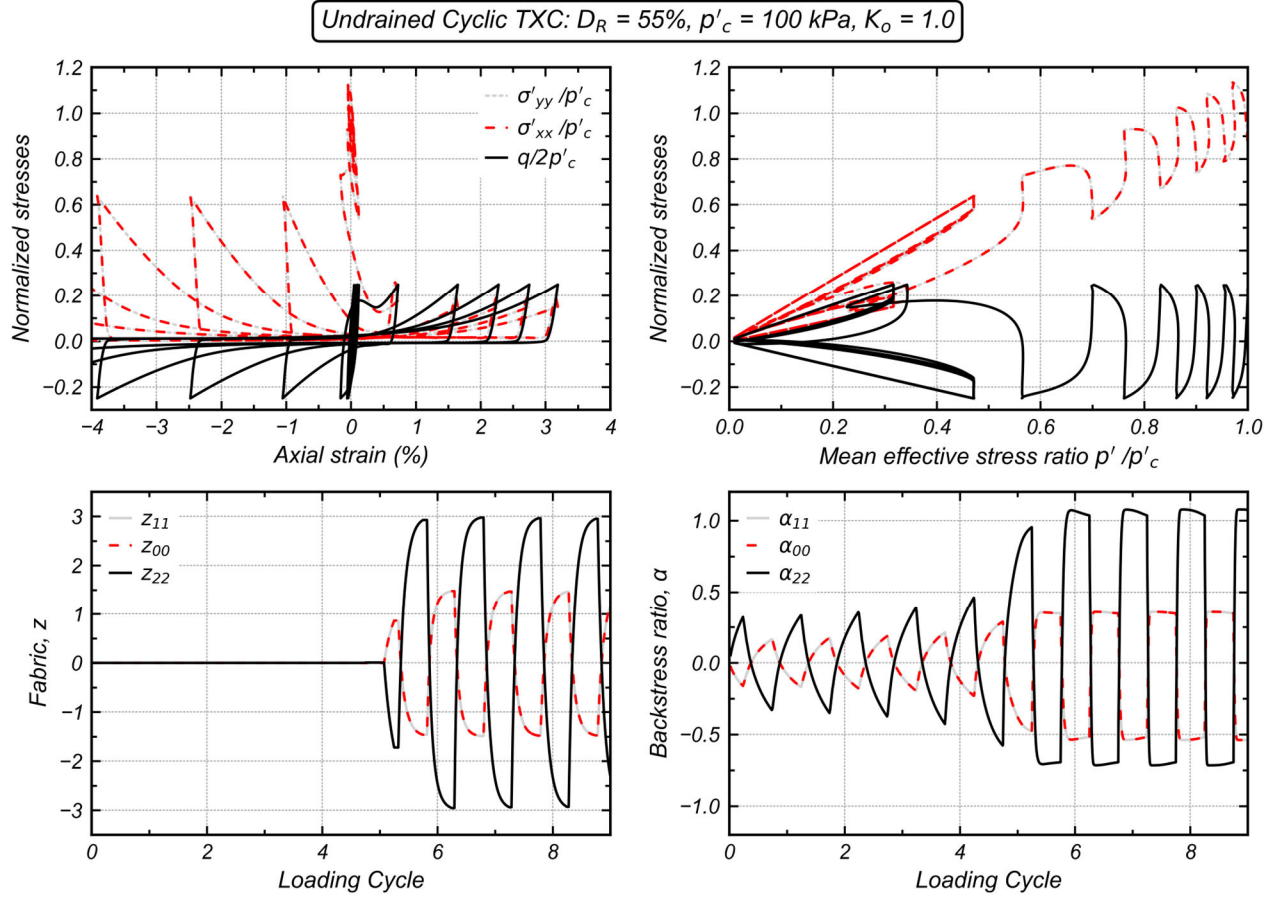


Figure 2.16. Undrained cyclic triaxial loading response for $D_R = 55\%$ with an initial static shear stress ratio of $\alpha=0.0$ (isotropically consolidated), showing the variation in stresses, back-stress ratios, and fabric tensor terms. Note that z corresponds to the vertical direction.

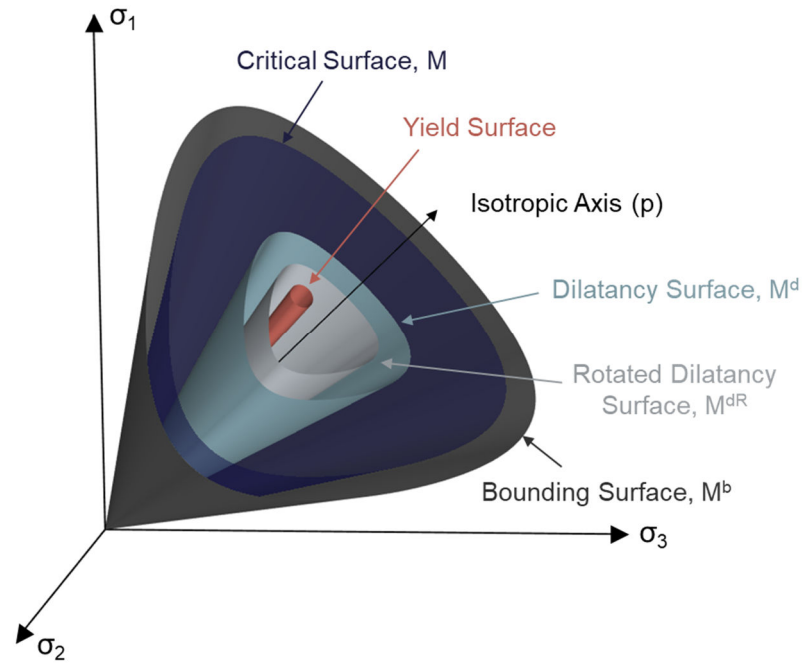


Figure 2.17. Schematic of the rotated dilatancy surface, along with the yield, critical, dilatancy, and bounding surfaces in principal stress space. Relative location of dilatancy and bounding surfaces corresponds to dense-of-critical states of stress.

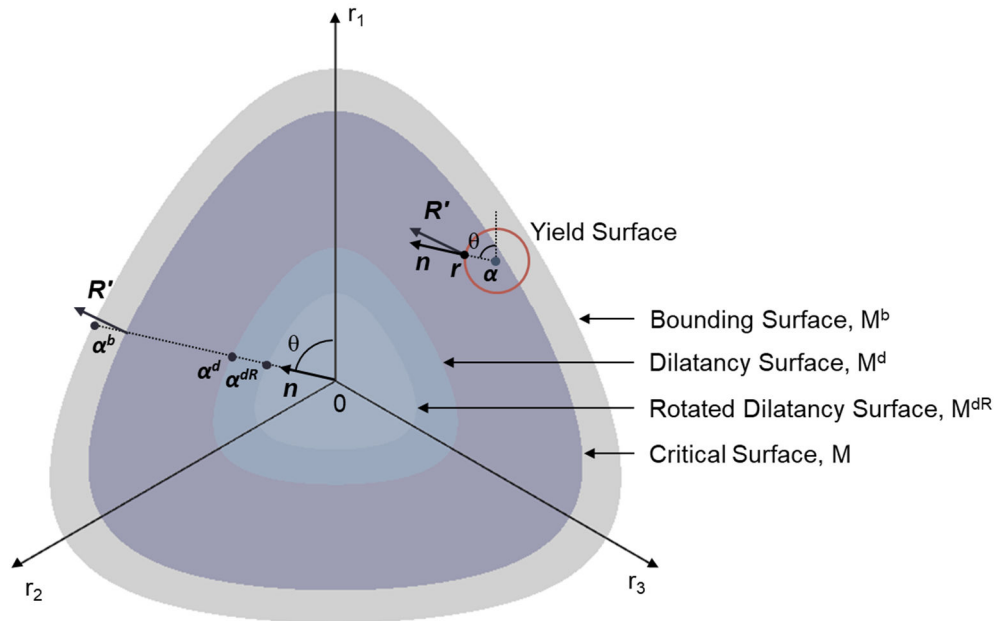


Figure 2.18. Schematic of the rotated dilatancy surface, along with the bounding, dilatancy, and critical surfaces on the stress ratio π plane with the yield surface, normal tensor, rotated dilatancy back stress ratio, dilatancy back stress ratio, and bounding back stress ratio (after Dafalias and Manzari 2004). Relative locations of the surfaces differ from those of Figure 2.17.

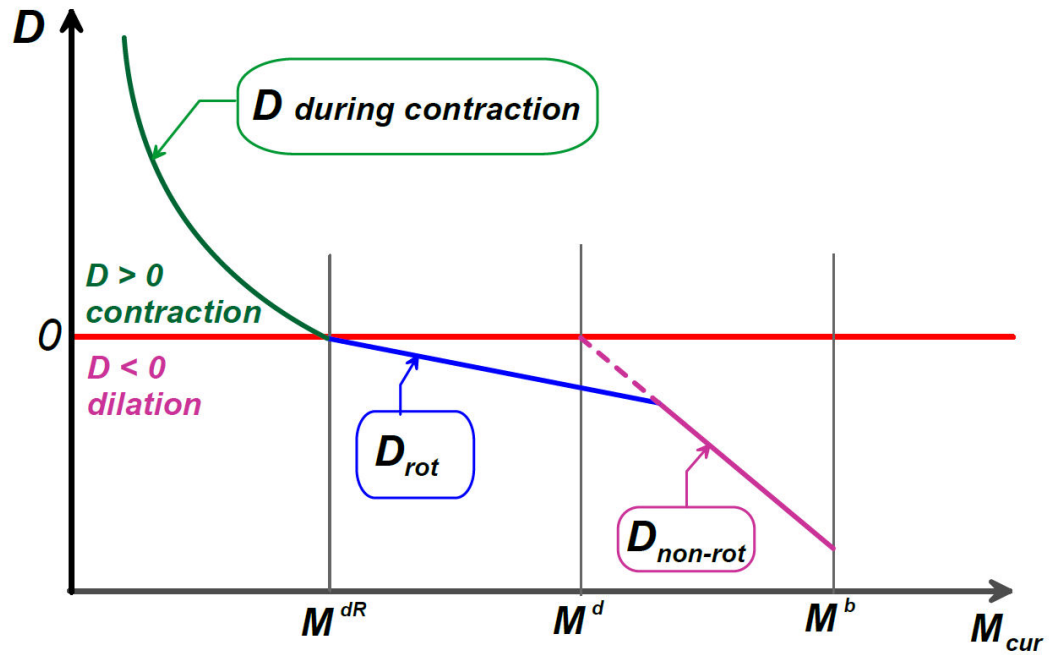


Figure 2.19. Schematic of the dilatancy D calculation based on the stress state with regards to the rotated dilatancy (M^{dR}), dilatancy (M^d) and bounding (M^b) surfaces during a half-cycle of loading that goes from contraction to dilation.

3. MODEL IMPLEMENTATION

The model has been implemented as a user-defined constitutive model (udm) for use with the commercial finite difference program, FLAC3D 9.0 (perpetual license) and 9.X (subscription license) (Itasca 2024). This section includes a brief description of the mixed discretization scheme used in FLAC3D, the numerical implementation scheme used for PM4Sand3D, some additional comments on alternative implementation schemes, and information regarding the dynamic link library (DLL) for PM4Sand3D.

3.1 Aspects of FLAC3D's numerical approach

Explicit integration

FLAC3D is an explicit finite difference commercial software which uses timesteps equal to or smaller than the minimum time required for waves to travel between any pair of nodes. This approach ensures that physical information does not propagate faster than numerical information. FLAC3D computes a default timestep based on the properties of the model (e.g., element size, material stiffness, permeability, damping). Users may specify a timestep that is smaller than the default value.

Obtaining numerically convergent solutions to nonlinear problems using FLAC3D requires that:

1. integration of the constitutive models is convergent, and
2. the explicit global solution is convergent.

The default timestep computed by FLAC3D does not necessarily ensure a numerically convergent solution, especially for FLAC3D models that are subjected to very high loading rates. Convergence of the constitutive model's integration depends more strongly on the strain increment size, which is dependent on both the loading rate and timestep size,

$$\frac{d\gamma}{dt} \cdot \frac{dt}{step} = \frac{d\gamma}{step} \rightarrow d\dot{\gamma} \cdot dt_{dyn} = d\gamma \quad (100)$$

Convergence of the explicit global solution depends more strongly on the sizes of the stress increments generated in the materials, which again are only indirectly controlled by the default timestep size. For this reason, the user needs to evaluate the sensitivity of the solution to the timestep size and not automatically assume that the default timestep size ensures a convergent solution.

Mixed discretization scheme

FLAC3D uses a mixed discretization technique in which each zone (analogous to an element) is subdivided internally by its diagonals into two overlaid sets of constant-strain tetrahedra. Note that the total number of tetrahedra in each zone depends on the selection of the basic zone type within FLAC3D's options, including, but not limited to, bricks, pyramids, and wedges. The term “mixed” stems from the fact that different discretizations are used for the isotropic and deviatoric parts of the strain and stress tensor (Marti and Cundall 1982). Isotropic stress and strain components are taken to be constant over the whole zone, while the deviatoric components are treated separately for each

tetrahedral sub-zone. Essentially, the shear strains are computed and maintained for each individual tetrahedral, while the volumetric strains are computed for each zone as a weighted average of the volumetric strains within the juxtaposed pairs of tetrahedra. Hourglass modes are resisted by shear stresses generated in the tetrahedra and the scheme accurately predicts plastic collapse loads because constant volume deformations are possible within the zones. Note that discretization using tetrahedra alone would result in meshes that are over-constrained (too stiff) and which would tend to overpredict plastic collapse loads. The symmetric response of this discretization can only be obtained by running two complete meshes in parallel, each representing one half of the overall stiffness. At the end, the procedure reduces the number of constraints on plastic flow and, at the same time, reduces unwanted hourglassing by ensuring that hourglass modes produce non-zero stresses.

The implementation of a complex constitutive model in FLAC3D requires special attention to the way stresses and strains are handled under FLAC3D's mixed discretization scheme. During each timestep, FLAC3D calls the constitutive model once per tetrahedra subzone. The isotropic components of the stress outputs from the subzones are then averaged internally by FLAC3D according to the mixed discretization scheme. A consequence of this averaging of stresses is that the final stress state for any subzone is unlikely to satisfy the consistency condition of elastoplastic models, meaning that the newly calculated stress states will not necessarily lie on the yield surface. Andrianopoulos et al. (2009) addressed this problem by adopting a vanished elastic region in their elastoplastic model.

3.2. Implementation of PM4Sand3D in FLAC3D

The implementation scheme for PM4Sand3D and how it relates to the challenges posed by FLAC3D's mixed discretization scheme (Section 3.1) are described here. Recall that each zone (consisting of an even number of subzones) will start off at the beginning of each timestep with a stress state and will be loaded by a strain increment whose volumetric components are the same in all the subzones while their deviatoric components are different (due to the mixed discretization scheme). The constitutive model will be called once per subzone to obtain stresses from strains according to the following equation where C_{ijkl} denotes the constitutive law:

$$\sigma_{ij}^{(i+1)} = \sigma_{ij}^{(i)} + d\sigma_{ij}^{(dt)} = \sigma_{ij}^{(i)} + C_{ijkl} d\epsilon_{ij}^{(dt)} \quad (101)$$

At the end of the step, each subzone will have its own stress state, which will be handled by FLAC3D independent of the constitutive model, and its own internal parameters, which FLAC3D will be unaware of. The subzones can therefore all have different stress states at the beginning of the next loading increment, and would need to maintain their own sets of internal parameters.

The current implementation scheme for PM4Sand3D is illustrated in Figure 3.1 and described by the pseudo-code listed in Table 3.1. At the end of each timestep, the stress and internal variables are averaged over the subzones. A drift correction is applied to ensure that the averaged stresses and internal variables satisfy the consistency condition; the correction involves projecting the back-stress ratio in the direction of the zone-averaged stress ratio. Another correction is applied if the zone-averaged stress ratio lies outside the bounding surface; the correction involves projecting the zone averaged stress ratio back towards the zero-stress state. The zone-averaged stresses are then used to compute a new dilatancy D and plastic modulus K_p that are consistent with the average response of the zone over this step. These values for D and K_p are then used by all subzones in the next timestep (i.e.,

the values of D and K_p lag one step behind the timestep for which they were determined); note that this approach is used by other elastoplastic models available in FLAC3D. Consequently, the subzones will use a common D and K_p during each timestep. Most other internal parameters are also computed and retained at the zone level, as described by the pseudo-code in Table 3.1.

The current implementation scheme of PM4Sand3D is based on iterations during the development of PM4Sand in plane-strain. Additional information on these iterations are described in Boulanger and Ziotopoulou 2017, PM4Sand Version 3.1 manual.

The current implementation also includes a scheme to reduce hourglassing modes which developed in liquefied zones in some cases because the subzones no longer have independent states of stress. The subzones have, in parallel to the PM4Sand3D constitutive model, an elastoplastic resistance to shear stresses which acts independently in each of the subzones. The properties of this parallel elastoplastic model are set at the instance when PM4Sand3D is initialized; the elastic moduli of the parallel elastoplastic model are set equal to 0.01 times those for PM4Sand3D, and its plastic shear strength (c_{hg}) is set as the product of a strength ratio (cr_{hg}) times the mean effective stress in the zone. If the user specifies values for both c_{hg} and cr_{hg} , then c_{hg} is taken as the greater of the specified c_{hg} value and the value computed using the specified cr_{hg} . The default value for cr_{hg} is 0.005. The parallel elastoplastic model only responds to deviatoric strains (producing shear stresses) and not to volumetric strains (producing no mean stress). This nominal amount of independent shearing resistance in the subzones was found to adequately control hourglassing modes for the range of problems examined to date.

Implementation of PM4Sand3D uses explicit integration and thus the user should routinely check that the solutions are not sensitive to timestep size. The addition of substepping (e.g. Sloan 1987) could improve the constitutive model's integration but would not eliminate the need to evaluate the effect of timestep size on the global solution. In our experiences, the default timesteps of FLAC3D in dynamic analyses of liquefaction problems have been small enough to ensure that numerical solutions are not significantly affected by timestep size, and thus the additional computational cost of including substepping at the constitutive level was not considered necessary. Examples of the effects of timestep size are presented in Section 3.3.

Some bounds have been placed on the values assigned to certain parameters at the time of initialization; e.g., the minimum value of mean stress is limited to 0.5 kPa or 0.005 times the initial consolidation stress; the relative density was limited to values less than 1.2. These initial bounds are to avoid simple mistakes like the user inputting relative density as a percentage.

3.3 Effect of timestep size on element responses

Numerical convergence of the current implementation of PM4Sand3D was evaluated by running numerous problems using a range of strain increment size under the default (maximum) timestep computed by FLAC3D. To achieve high velocities, the timestep may need to be decreased from the default timestep. The user must always check the sensitivity of boundary value problems to the timestep size and strain increment size.

For example, the effect of strain increment size ($\Delta\gamma$) on integration of the PM4Sand3D model is shown in Figure 3.2(a) for a single element simulation of a cyclic drained DSS test for sand at $D_R = 55\%$ at $\sigma'_{vo} = 100\text{kPa}$. The element was subjected to two cycles of strain-controlled loading with a

single-amplitude shear strain of 1%. The default timestep was $5.04\text{e-}4$ seconds (per step). A range of strain increment sizes were explored ranging from $6.30\text{e-}5$ %/step to $5.04\text{e-}4$ %/step. Therefore, the range of resulting strain rates were 0.125%/sec to 1%/sec. The simulated stress-strain responses showed minimal differences, indicating that the integration was sufficiently accurate for practical purposes at these strain increment sizes.

A second example of the effect of strain increment size is shown in Figure 3.2(b) for a stress-controlled cyclic undrained DSS test for sand at $D_R = 55\%$ at $\sigma'_{vo} = 100\text{kPa}$. The default timestep was $7.0\text{e-}5$ seconds. The default timestep is smaller for the undrained element test because of the higher wave speed in the pore water. The same range of strain rates was explored, ranging from 0.125%/sec to 1%/sec. Due to the smaller default timestep, the range in loading strain increment sizes thus ranged from $8.75\text{e-}6$ %/step to $7\text{e-}5$ %/step. Small differences were observed within this range of strain increments; Figure 3.2(b) demonstrates the need for the user to examine their results at varying strain increment size whether controlled directly or chosen indirectly due to the chosen timestep and applied velocities.

Note that comparisons of solutions at different step sizes $\Delta\gamma$ generally cannot be made by just varying the strain rate or cyclic loading frequency. FLAC3D is always solving the dynamic equation of equilibrium so changing the strain rate by changing any loading rate parameter also changes the dynamic excitation for the system, which can cause a change in the dynamic response of the element. In that case, any changes in the stress-strain response caused by changes in loading rate parameters may be a realistic simulation result that reflects the change in the dynamic excitation of the element.

3.4 DLL module

The PM4Sand3D constitutive model was coded in C++ and compiled as a DLL in Microsoft Visual Studio 2019. It has been tested in FLAC3D 9.0/9.X using the software's option for User Defined Models (UDMs). The steps required for using a DLL are described in the FLAC3D manuals (Itasca 2024) and are thus only briefly summarized herein.

Automatic loading of the DLL file

1. Load the DLL file in the /exe64/plugins/cmodel subdirectory of the folder where FLAC3D has been installed.
2. Open the FLAC3D 9.0/9.X executable file or the FLAC3D graphical user interface. If the DLL is properly located, then the model should be automatically loaded. Upon opening the FLAC 3D graphical user interface, the console will list the user defined models that have been loaded into the program. Available constitutive models can also be listed using the zone cmodel list command.
3. Before constitutive model plug-ins can be assigned to zones, the model must be configured for their use by giving the model configure plugin command. Otherwise, the user will get the following error message: "Not configured for user defined constitutive model plugins. See the MODEL CONFIG command."

Manual loading of the DLL file

1. Load the DLL file in the /exe64/plugins/cmodel subdirectory of the folder where FLAC3D has been installed.
2. In the project file (or the called fish file for the analysis) the model must be first configured for the use of constitutive model plug-ins (`model configure plugin`) and then the model's DLL

can be loaded (program load cmodel "plugins/cmodel/cmodelPM4Sand3D009.dll"). The console will provide a message confirming the plugin file has been loaded. The available constitutive models can also be listed to the console (zone cmodel list).

To assign the model to all elements, the following command should be given:

```
zone cmodel assign pm4sand3d
```

Additional notes regarding the DLL file

The automated ability to use the DLL with FLAC3D's "free-field" lateral boundary conditions option has not been configured at this time. However, the user can use the "free-field" boundary by manually mapping/assigning the PM4Sand3D model to the free-field zones. This task can be completed efficiently using a fish function (e.g., an example function to do this FLAC2D is posted online). PM4Sand3D also cannot be used in the bottom row of elements above a compliant base because it has not been configured to automatically transfer appropriate elastic moduli to the compliant base.

3.5 Additional notes on use in boundary value problem simulations

FLAC3D includes both "static" and "dynamic" solution procedures. Validation studies for PM4Sand3D have focused on the dynamic procedure. The use of PM4Sand3D with the static solution procedure requires special attention to the loading and solution procedures. The static solution procedure uses extremely high damping values which can carry significant shear and normal stresses, which can cause problems with the response of a highly nonlinear, stress-dependent material. For example, if the user imposes a large strain rate (e.g., high rate of loading on a foundation) in a problem involving drained loading of a contractive soil with the static solution procedure, the drained volumetric contraction of the soil can result in normal stresses being transferred to the damping component which causes an artificial reduction in normal effective stress in the soil. For this reason, the use of PM4Sand with FLAC's static solution procedure requires scrutiny and evaluation to ensure that such problems do not develop.

A nominal amount of Rayleigh damping needs to be included with PM4Sand3D zones to control numerical noise. A damping ratio of 0.005 has been found sufficient for most applications.

Experience in PM4Sand has shown loading conditions that cause a progressive increase in the mean effective stresses in PM4Sand, or any other pressure-dependent material, including PM4Sand3D, require special consideration during the solution process. The elastic moduli will increase with increasing mean effective stress, such that the timestep required for a stable solution will decrease as the loading progresses. FLAC3D only determines the required timestep at certain instances, like when the step or solve commands are executed. For this reason, the loading should be applied in small increments with the solve command periodically repeated so that the required timestep is updated as appropriate during the applied loading.

Initial stresses in a boundary value problem are sometimes established using simpler constitutive models, like a Mohr-Coulomb or elastic model, prior to switching the materials to a more complex model like PM4Sand3D. Issues can arise if the initial states of stress fall outside the bounding surface lines for the PM4Sand3D model. This can happen in zones where the initial state of stress was computed for a Mohr-Coulomb material with a nonzero cohesion or for an elastic material. For this

reason, it is helpful to first ensure that the initial states of stress in all zones correspond to a stress ratio that is less than some reasonable limit prior to switching the material model to PM4Sand3D.

Table 3.1: Simplified pseudo-code of PM4Sand3D (Version 1)

Operations within one subzone:

1. Initialize the model parameters; this only happens when the model is first assigned or when FirstCall is set to zero at some point during the analysis. For detailed information on what parameters are initialized (or reset) see Table 3.2.
2. Obtain the strain increment from FLAC3D $d\boldsymbol{\varepsilon}$.
3. Obtain rotational velocities from FLAC3D for use in large-strain simulations.
4. Decompose the strain increment into volumetric and deviatoric components, $d\boldsymbol{\varepsilon}_p$ and $d\boldsymbol{\varepsilon}_s$.
5. Calculate the trial elastic stress increment and trial elastic stress:

$$\boldsymbol{\sigma}_{tr} = \boldsymbol{\sigma}_0 + d\boldsymbol{\sigma}_{tr} = \boldsymbol{\sigma}_0 + 2Gd\boldsymbol{\varepsilon} + Kd\boldsymbol{\varepsilon}_v \mathbf{I}$$

6. Calculate the trial stress ratio r_{tr} , the distance from the yield surface $dist$, the unit normal to the yield surface $\mathbf{n}$, and the inner product of the change in back-stress ratio tensor with unit normal vector $d\boldsymbol{\alpha} : \mathbf{n}$.

$$r_{tr} = \frac{\boldsymbol{\sigma}_{tr} - p_{tr} \mathbf{I}}{p_{tr}}$$

$$dist = \sqrt{(\mathbf{r}_{tr} - \boldsymbol{\alpha}_0) : (\mathbf{r}_{tr} - \boldsymbol{\alpha}_0)}$$

$$\mathbf{n} = \frac{(\mathbf{r}_{tr} - \boldsymbol{\alpha}_0)}{dist}$$

$$d\boldsymbol{\alpha} : \mathbf{n} = (\boldsymbol{\alpha}_0 - \boldsymbol{\alpha}_{in}) : \mathbf{n}$$

7. Check for yield:
 - a. If elastic then commit the trial stresses. Go to step 8.

$$dist < \sqrt{\frac{2}{3}} m$$

$$\boldsymbol{\sigma}_0 = \boldsymbol{\sigma}_{tr}$$

- b. If inelastic:
 - i. Calculate loading index L :

$$L = \frac{2G\mathbf{n} : d\boldsymbol{\varepsilon} - \mathbf{n} : \mathbf{r} K d\boldsymbol{\varepsilon}_v}{K_p + 2G(B - C \cdot tr \mathbf{n}^3) - K D \mathbf{n} : \mathbf{r}}$$

- ii. Calculate trial stress increment and trial stress:

$$\boldsymbol{\sigma}_{tr} = \boldsymbol{\sigma}_0 + d\boldsymbol{\sigma}_{tr} = \boldsymbol{\sigma}_0 + 2Gd\boldsymbol{\varepsilon}_s + Kd\boldsymbol{\varepsilon}_p \mathbf{I} - \langle L \rangle \left(2G \left[B \mathbf{n} \left(\mathbf{n}^2 - \frac{1}{3} \right) \mathbf{I} \right] + K D \mathbf{I} \right)$$

- iii. Calculate image back-stress ratios and inner products:

$$\boldsymbol{\alpha}^b = \sqrt{\frac{2}{3}} [M^b - m] \mathbf{n}$$

$$\boldsymbol{\alpha}^d = \sqrt{\frac{2}{3}} [M^d - m] \mathbf{n}$$

$$\boldsymbol{\alpha}^{dR} = \sqrt{\frac{2}{3}} [M^{dR} - m] \mathbf{n}$$

Table 3.2: Simplified pseudo-code of PM4Sand3D (Version 1) (cont.)

- iv. Calculate trial back-stress increment and trial back-stress:

$$\alpha_{tr} = \alpha_0 + d\alpha_{tr} = \alpha_0 + \frac{\langle L \rangle K_p}{p_0} \cdot \frac{\alpha^b - \alpha_0}{(\alpha^b - \alpha_0 : \mathbf{n})}$$

- v. Apply penalties to stress ratios and back-stress ratios to meet the consistency condition and to remain within the bounding surface.
- vi. Commit the trial stresses (back-stress ratios, stress ratio, mean stress, stress)
8. Return all stress tensor components to FLAC3D (at this point FLAC3D takes over and will average them according to the mixed discretization scheme).

Operations referring to the whole zone:

9. After the calculation has completed the final subzone, the following additional calculations are performed for the overall zone. Recall the following parameters for all subzones and compute area-weighted average values for:
- Volumetric strain $d\bar{\varepsilon}_p$
 - Strain increment $d\bar{\varepsilon}$
 - Mean stress $\bar{p}$
 - Stress tensor (committed one) $\bar{\sigma}_0$
 - Back-stress ratio tensor $\bar{\alpha}_0$
 - Unit normal to yield surface vector $\bar{\mathbf{n}}$
 - Rotational velocities
10. Apply penalties to the averaged zone parameters to meet the consistency condition and maintain the yield surface inside the larger of the bounding and dilatancy surfaces.
11. Calculate image back-stress ratios and inner products for the averaged zone parameters.
12. Calculate daxn for the averaged zone parameters and determine whether a loading reversal has occurred.
13. Compute dilatancy, D , and plastic modulus, K_p , for the past average step in the zone.
14. Compute plastic volumetric strain for use in fabric terms.
15. If $(\alpha^d - \alpha) : \mathbf{n} < 0$, update the fabric tensor for the zone. and if exceeding its former value, update the cumulative fabric term.

$$\mathbf{z} = \mathbf{z} - \frac{c_z}{\max\left(1, \frac{z_{cum}}{2\sqrt{2}z_{max}}\right)} \frac{d\bar{\varepsilon}_v^{pl}}{D} (z_{max}\bar{\mathbf{n}} + \mathbf{z})$$

16. Update related fabric terms including z_{cum} , if fabric exceeds former value, peak fabric, z_{pk} , the product of p and z , p_{zp} (and associated $p_{zp,peak}$ when product exceeds former values), and upon reversals: z_{in} .
17. Calculate new g based on Lode angle calculated based upon the unit normal to yield surface ($\bar{\mathbf{n}}$) to be applied to bounding and dilatancy stress ratios.
18. Update the relative state parameter, the bounding and dilatancy stress ratios, the elastic shear modulus (depends on fabric) and the elastic bulk modulus for the next step.
19. Update the initial and previous initial back-stress values and the strain increment accumulators.
20. Update initial back-stress ratios upon reversal.
21. Commit zone stress tensor, zone mean stress, zone back-stress ratio tensor, zone stress-ratio tensor to memory.
22. If working in FLAC3D's large-strain mode, apply rotation corrections to zone stress, zone back-stress ratio, zone stress-ratio, and zone fabric tensors.
23. If using post-shaking adjustments, reduce the post-shaking elastic shear modulus G and elastic bulk modulus K .

Table 3.3: Initialization function of PM4Sand3D (called during the first application of the model and whenever FirstCall=0)

1. Obtain stresses from FLAC3D and create stress tensor (these will be the committed stresses from which the calculation will start): σ_o^{ij}

2. Check stresses and calculate mean effective stress:

a. If stresses compressive (following FLAC3D's sign convention that tensile stresses and strains are positive):

$$\sigma_o^{11} < 0 \rightarrow p_0 = \min\left(p_{min}, \frac{1}{3}\sigma_o^{ij}\right)$$

$$p_{min} = \min\left(p_{min}, \frac{p}{200}\right)$$

$$p_{min2} = \min\left(p_{min2}, \frac{p}{20}\right)$$

b. If stresses tensile:

$$\sigma_o^{11} > 0 \rightarrow p_0 = -\frac{P_{atm}}{20}$$

$$\sigma_o^{ij} = p_0 \mathbf{I}$$

3. Calculate relative state parameter and initial Lode angle (based on stress invariants) and then subsequently calculate the bounding and dilatancy stress ratios and A_{do} (from input property D_R and secondary parameters R , Q , n^b and n^d – see Section 4):

$$\xi_R = \frac{R}{Q - \ln\left(100 \frac{p}{p_A}\right)} - D_R$$

$$\cos(3\theta) = \frac{3\sqrt{3}}{2} \frac{\det(\sigma - p\mathbf{I})}{\left(\frac{1}{2}(\sigma - p\mathbf{I} : \sigma - p\mathbf{I})\right)^{3/2}}$$

$$g(\theta, c) = \frac{2^{1/4} c}{\left[(1 + c^4) - (1 - c^4) \cos(3\theta)\right]^{1/4}}$$

c. If dense-of-critical ($\xi_R < 0$):

$$M^b = M \cdot g \cdot \exp(-n^b \xi_R) \quad M^d = M \cdot g \cdot \exp(n^d \xi_R)$$

$$A_{do} = 3.125 \frac{\left[\sin^{-1}\left(\frac{3M^b}{6+M^b}\right) - \sin^{-1}\left(\frac{3M}{6+M}\right)\right]}{(M^b - M^d)}$$

d. If loose-of-critical ($\xi_R > 0$):

$$M^b = M \cdot g \cdot \exp\left(-\frac{n^b}{4} \xi_R\right) \quad M^d = M \cdot g \cdot \exp(4n^d \xi_R)$$

$$A_{do} = 1.04$$

4. Check that initial stresses are inside the bounding surface (or dilatancy surface if it is greater) and compute the committed back-stress and stress ratio tensors from the stress tensor:

$$M^{cut} = \max(M^b, M^d)$$

$$M^{fin} = -\frac{1}{p_0} \sqrt{\frac{3}{2} \left((\sigma_0^{ij} - p_0 I) : (\sigma_0^{ij} - p_0 I) \right)}$$

- a. If $M^{fin} > M^{cut}$:

$$r_0^{ij} = \frac{(\sigma_0^{ij} - p_0 I)}{p_0} \left(\frac{M^{cut}}{M^{fin}} \right)$$

$$\sigma_0^{ij} = p_0 I + r_0^{ij} p_0$$

$$\alpha_0^{ij} = r_0^{ij} \left(\frac{M^{cut} - m}{M^{cut}} \right)$$

- b. If $M^{fin} < M^{cut}$:

$$r_0^{ij} = \frac{(\sigma_0^{ij} - p_0 I)}{p_0}$$

$$\alpha_0^{ij} = r_0^{ij}$$

5. Create/Initialize the initial back-stress ratio, initial previous back-stress ratio, minimum initial back-stress ratio and maximum initial back-stress ratio tensors (see also Section 2.5 on Stress Reversal):

$$\alpha_{inP}^{ij} = \alpha_{inMax}^{ij} = \alpha_{inMin}^{ij} = \alpha_{in}^{ij} = \alpha_0^{ij}$$

6. Calculate initial values of elastic shear modulus, elastic bulk modulus, plastic modulus, dilatancy:

$$G = G_o p_A \left(\frac{-p_0}{p_A} \right)^{1/2}$$

$$K = \frac{2(1+\nu)}{3(1-2\nu)} G$$

$$K_p = 100G$$

$$D = 0$$

7. Initialize fabric related terms (see Section 2.8) – note that these terms will be referring to the whole zone:

$$p_{zp} = \frac{p_0}{100}$$

$$z_{peak} = \frac{z_{max}}{100000}$$

$$zxp = z^{ij} : p = 0$$

$$zxpPk = -z_{max} \frac{p_0}{50}$$

$$z^{ij} = z_{in}^{ij} = z_{\alpha}^{ij} = z_{cum} = 0$$

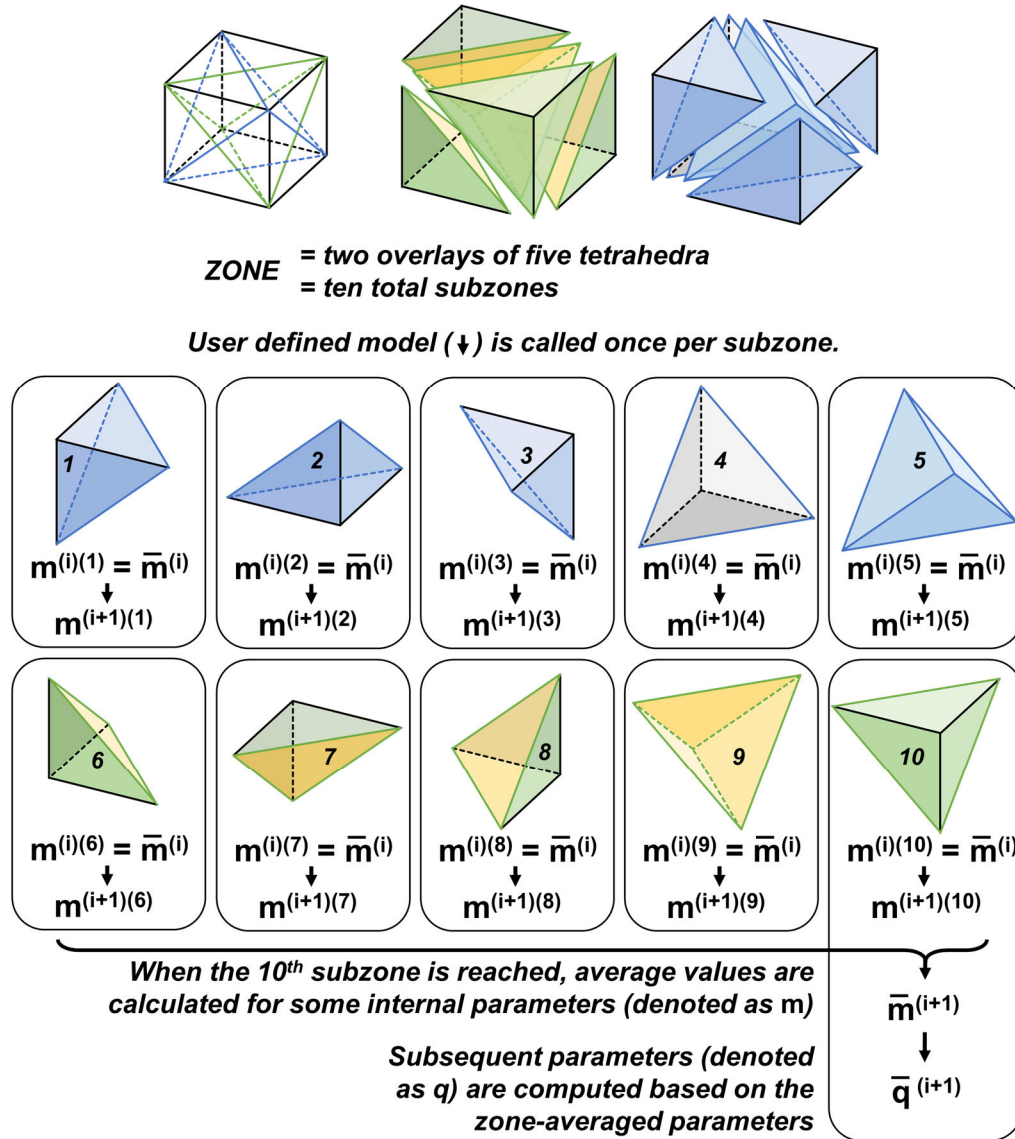


Figure 3.1. Schematic illustration of the averaging procedure followed in the implementation of PM4Sand3D: zone-averaged values are computed for some internal variables of the model, denoted as “m”, at the end of each step, after which other internal parameters, denoted as “q”, are computed based on the zone-averaged parameters. Note that this schematic represents the “brick” zone type in FLAC3D. Additional zone shapes are possible; these zone shapes may have fewer subzones.

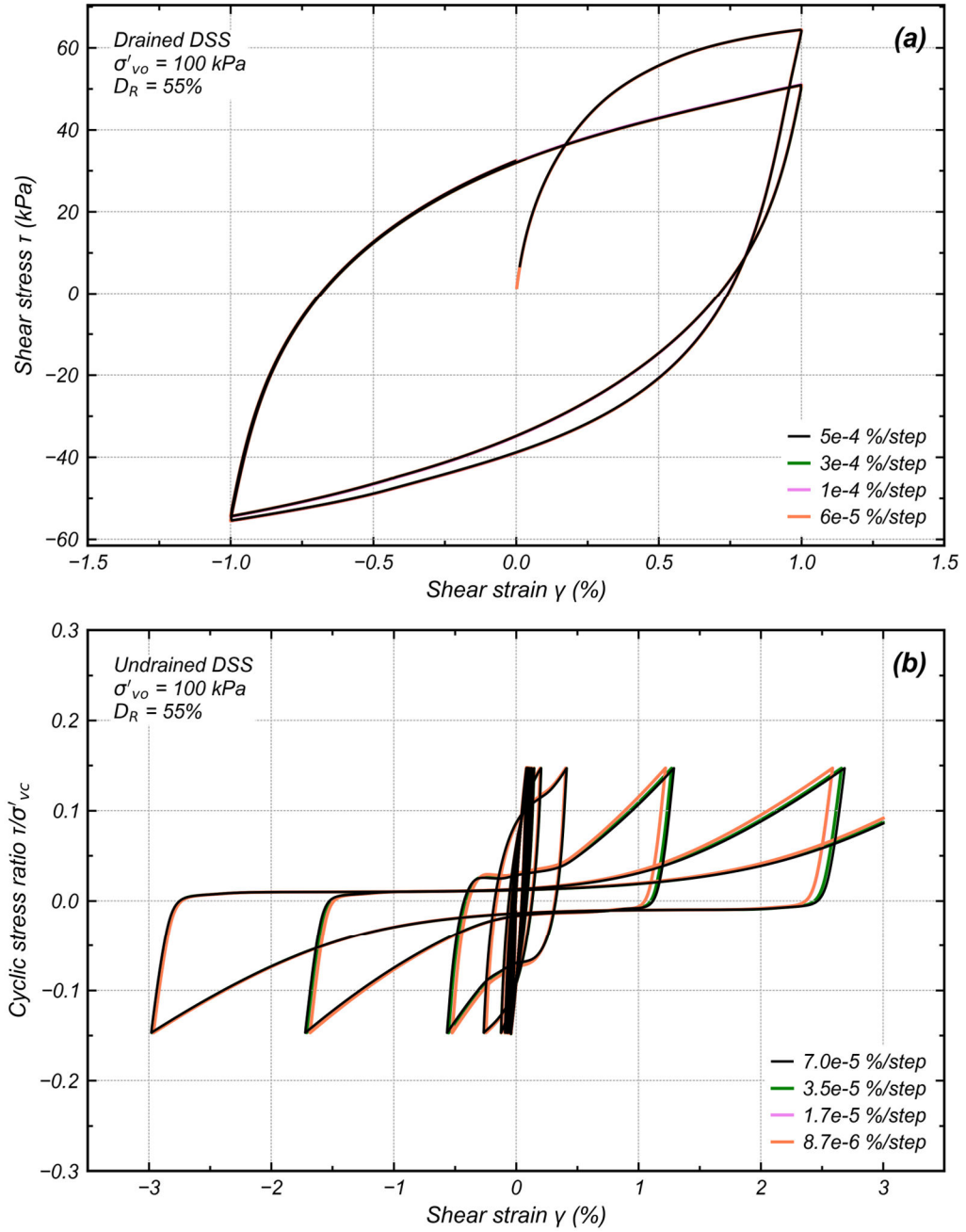


Figure 3.2. Effect of dynamic timestep on the results obtained from (a) drained and (b) undrained cyclic DSS element test simulations ($D_R=55\%$, $\sigma'_{vc}=1\text{atm}$) loaded at a shear strain rate of 1%/sec. The black line in each case denotes the response obtained with FLAC3D's default dynamic timestep.

4. MODEL INPUT PARAMETERS AND RESPONSES

4.1 Model input parameters

The model parameters are grouped into two categories; a primary set of six parameters (three properties, two flags, and atmospheric pressure) that are most important for model calibration, and a secondary set of parameters that may be modified from their default values in special circumstances.

Primary input parameters

The three primary input properties are the sand's apparent relative density D_R , the shear modulus coefficient G_o , and the contraction rate parameter h_{po} . These three parameters are discussed below and summarized in Table 4.1.

The apparent relative density (D_R) can be estimated in practice by correlation to penetration resistances. For example, a common form for SPT correlations is,

$$D_R = \sqrt{\frac{(N_1)_{60}}{C_d}} \quad (102)$$

where D_R is expressed as a ratio rather than a percentage. Idriss and Boulanger (2008) reviewed published data and past relationships and then adopted a value of $C_d=46$ in the development of their liquefaction triggering correlations. For the CPT, they similarly reviewed available relationships and arrived at the following expression,

$$D_R = 0.465 \left(\frac{q_{c1N}}{C_{dq}} \right)^{0.264} - 1.063 \quad (103)$$

for which they adopted $C_{dq}=0.9$. For the example loading responses shown later, D_R values of 35%, 55%, and 75% were used, which would correlate to SPT $(N_1)_{60}$ values of 6, 14, and 26 by the above correlations.

The input value of D_R is best considered an "apparent relative density," rather than a strict measure of relative density following conventional laboratory tests. The input value of D_R influences the response of the model and thus it is just another input parameter that the user can adjust as part of the calibration process. The above correlations are provided for the purpose of obtaining a reasonable estimate for the apparent D_R so that the resulting model behaviors are also reasonable. There are situations, however, where the user may choose to adjust the input D_R , up or down relative to the above relationships, to improve its calibration to some other relationship or data.

The second primary input parameter is the constant G_o which will control the elastic (or small strain) shear modulus. The equation for the elastic shear modulus at the time of model initialization can be simplified to,

$$G = G_o p_A \left(\frac{p}{p_A} \right)^{1/2} \quad (104)$$

The elastic shear modulus can be calibrated to fit in-situ V_s measurements, according to,

$$G = \rho \cdot (V_s)^2 \quad (105)$$

or alternatively fit to values of V_s that may be estimated by correlation to penetration resistances. For the examples shown herein, the correlation by Andrus and Stokoe (2000) in Figure 4.1 was used, with a slight modification that constrains the extrapolation to very small $(N_1)_{60}$ values, as shown in the figure,

$$V_{sI} = 85 \left[(N_1)_{60} + 2.5 \right]^{0.25} \quad (106)$$

The above relationships in combination with the default values for the maximum and minimum void ratios (described later) produce V_{sI} values of 145, 171, and 196 m/s and corresponding G_o values of 506, 717, and 961 for the D_R values of 35%, 55%, and 75%, respectively.

Alternatively, the above expressions were combined with a range of typical densities to arrive at the following simpler expression for estimating G_o ,

$$G_o = 177 \sqrt{(N_1)_{60} + 2.5} \quad (107)$$

This expression produces G_o values of 504, 717, and 943 for the D_R values of 35%, 55%, and 75%, respectively.

The third primary input parameter is the constant h_{po} which is used to modify the contractiveness and hence enable calibration of the model to specific values of cyclic resistance ratio (CRR). For the examples presented herein, the target CRR values were based on the liquefaction triggering correlation by Idriss and Boulanger (2008) in Figure 1.3. This relationship produces target CRR values for an effective overburden stress of 1 atm and an earthquake magnitude of $M=7.5$ of 0.090, 0.147, and 0.312 for the corresponding SPT $(N_1)_{60}$ values of 6, 14, and 26, respectively.

The value of atmospheric pressure, p_A , should also be specified in the unit set being used for the analysis. If not specified, it will default to 101,300 Pascal.

The flag FirstCall is used to re-set the back-stress ratio history terms equal to the current stress ratio, and to erase all fabric terms. The first time the model is called, the flag should be unspecified or have a value of 0. The model will then initiate the back-stress ratios and all pertinent history terms using the current state of stress. The flag is then set equal to 1.0 internally. If FirstCall is later set equal to 0.0 using the property command in FLAC3D, this will cause the material to re-initiate all internal terms, thereby re-setting the back-stress and stress ratio history terms and erasing all fabric terms. FirstCall

should usually be set to 0.0 just before initiating dynamic earthquake loading. Otherwise, the model will retain memory of the loading during the static initiation of the model, which may or may not be desired.

The flag PostShake is used during the post-shaking portion of a simulation to improve the modeling of post-liquefaction reconsolidation strains. The flag is set to 0 internally and remains 0 unless the user specifies otherwise. If the flag is set to 1.0, the elastic moduli will be reduced according to the expressions presented previously. PostShake should only be set to 1.0 at the end of strong shaking, as the reductions in elastic moduli were not calibrated for cyclic loading behavior.

Table 4.1. Primary input parameters (parameter names in square brackets correspond to the input name to be used within FLAC3D)

Parameter [FLAC3D property name]	Comments
D_R [D_r]	Apparent relative density: Primary variable controlling dilatancy and stress-strain response characteristics. Input as a fraction, not as a percentage. Commonly estimated based on CPT or SPT penetration resistances, such as the following relationships used by Idriss and Boulanger (2008): $D_R = \sqrt{\frac{(N_1)_{60}}{C_d}} \quad \text{with } C_d = 46, \text{ and}$ $D_R = 0.465 \left(\frac{q_{clN}}{C_{dq}} \right)^{0.264} - 1.063 \quad \text{with } C_{dq} = 0.9.$
G_o [G_o]	Shear modulus coefficient: Primary variable controlling the shear modulus, G_{max}. Should be chosen to match estimated or measured shear wave velocities according to $G_{max} = \rho V_s^2$. A value for G_o can be estimated based on the modified correlation between SPT $(N_1)_{60}$ and V_{sl} values shown in Figure 4. The value of G_o can thus be computed as, $G_o = 177 \sqrt{(N_1)_{60}} + 2.5$
h_{po} [h_po]	Contraction rate parameter: Primary variable that adjusts contraction rates and hence can be adjusted to obtain a target cyclic resistance ratio, as commonly estimated based on CPT or SPT penetration resistances and liquefaction correlations. Calibration of this parameter should be performed last because its value can depend on the values assigned to other parameters.
p_A [P_atm]	Atmospheric pressure in the unit set being used. Defaults to 101,300 Pascals if not specified.
FirstCall [First_Call]	Flag used to reset the back-stress ratio history terms equal to the current stress ratio, and to erase all fabric terms. FirstCall should usually be set to 0.0 at model initiation and/or just before initiating dynamic earthquake loading. Otherwise, the model will retain memory of the loading during the static initiation of the model, which may or may not be desired.
PostShake [Post_Shake]	Flag used during post-shaking portion of a simulation to improve modeling of post-liquefaction reconsolidation strains. PostShake should only be set to 1.0 after the end of strong shaking.

Secondary input parameters

Secondary input parameters are those parameters for which default values have been developed that will generally produce reasonable agreement with the trends in typical design correlations. The user must, however, still confirm through element loading calibrations that the default parameters are appropriate for their particular conditions. The secondary input parameters (23 in total) are listed in Table 4.2, along with commentary on the recommended default values.

The recommended default values for all secondary parameters have been embedded within the initialization section of the code and thus these parameters will take their default values unless the user specifies otherwise. In addition, the input logic is structured such that secondary parameters will take their default value if the user inputs a value of zero for that parameter.

Table 4.2. Secondary input parameters

Parameter [FLAC3D name]	Comments
h_o [h_o]	Variable that adjusts the ratio of plastic modulus to elastic modulus. The default value of $h_o = (0.25 + D_R)/2$, with a minimum value of 0.30, was chosen to provide reasonable G/G_{max} and damping relationships for the default value of G_o . This variable may require adjustment in combination with any adjustments to G_o .
e_{max} and e_{min} [e_max] [e_min]	The maximum and minimum void ratios affect the computation of density, as well as how volumetric strains translate into changes in relative state. Default values of 0.8 and 0.5, respectively, were adopted. Refinements in these parameters for a practical problem may not be necessary, as the calibration of other parameters will have a stronger effect on monotonic or cyclic strengths.
n^b [n_b]	Default value is 0.50. Controls dilatancy and thus also the peak effective friction angles. Note that M^b for loose-of-critical states is computed using $n^b/4$.
n^d [n_d]	Default value is 0.10. Controls the stress-ratio at which contraction transitions to dilation, which is often referred to as phase transformation. A value of 0.10 produces a phase transformation angle slightly smaller than ϕ_{cv} , which is consistent with experimental data. Note that M^d for loose of critical states is computed using $4n^d$.
A_{do} [A_do]	Default value is computed based on Bolton's dilatancy relationship at the time of initialization; typical values will be between 1.04 and 1.09.
z_{max} [z_max]	Default value is computed at the time of initialization as, $z_{max} = 0.70 \cdot \exp(-6.1 \cdot \xi_{Ro}) \leq 20$ This returns 0.7 if ξ_R is initially 0.0 and increases to its maximum value of 20 with increasing dense-of-critical states. May require varying if the relationship between D_R and cyclic strength is significantly different from that implied by the liquefaction correlations of Idriss and Boulanger (2008).

Table 4.2. Secondary input parameters (cont.)

C_z [c_z]	Default value varies with D_R . The value is 100 for D_R less than 35%, and linearly increases to a value of 400 at $D_R = 75\%$. Controls strain levels at which fabric affects become important and can be used to adjust the rate of strain accumulation in undrained cyclic loading.
C_ε [c_e]	Default value is computed at the time of initialization as, $C_\varepsilon = 50e^{-11.5 \cdot (D_{Ro} - 0.35)} \geq 0.3$ Can be used to adjust the rate of strain accumulation in undrained cyclic loading.
ϕ'_{cv} [phi_cv]	Default value is 32 degrees. This is the value for triaxial compression; values for other Lode angles depend on the parameter LA_c .
ν_o [pois]	Default value is 0.30. For 1-D consolidation of an elastic material, the value of K_o would correspond to, $K_o = \frac{\nu}{1-\nu}$ The default value for ν results in a K_o value of 0.43 in 1-D consolidation.
C_{GD} [G_degr]	Default value is 2.0. The small-strain elastic modulus degrades with increasing cumulative plastic deviator strains (z_{cum}). The maximum degradation approaches a factor of $1/C_{GD}$.
C_{DR} [C_DR]	Default value is computed at the time of initialization as, $C_{DR} = 5 + 25(D_{Ro} - 0.35) \leq 10$ Controls the rotated dilatancy surface and is applied to reduce the rate under which dilatancy is increasing.
C_{Kaf} [C_kaf]	Default value varies with D_R . as, $C_{Kaf} = 1 + 600 \cdot (D_{Ro} - 0.37)^3 \in [3.0; 50.0]$ The value is 3.0 for D_R less than 10% and increases to its maximum value of 50.0 at $D_R = 80\%$. This variable controls the effect that sustained static shear stresses have on plastic modulus.
n_g [ng]	Default value varies with D_R as, $n_g = 0.5(2.35 - D_{Ro}) \leq 1.0$ This variable controls the amount of bias towards extension (versus compression) in undrained cyclic loading.
c [LA_c]	Default value computed at the time of initialization, based on ϕ'_{cv} , as, $c = 0.9I - 0.15M \geq 0.64$ Relationship chosen to maintain reasonable friction angles across range of loading directions.
Q [Q_bolt]	Default value is 10. Default value is for quartzitic sands per recommendations of Bolton (1986).

Table 4.2 – Secondary input parameters (cont.)

R [R_bolt]	Default value is 1.5. Default value for quartzitic sands would be 1.0 per recommendations of Bolton (1986); a slight increase in R is used to lower the critical state line to better approximate typical results for direct simple shear loading.
m [m_par]	Default value is 0.01. Default value provides reasonable modeling and numerical stability.
F _{sed,min} [F_sed]	Default value is. computed at the time of initialization as a function of relative density: $F_{sed,min} = 0.04 + 0.0625(D_R - 0.35) \leq 0.99$ Controls the minimum value the reduction factor of the elastic moduli can attain during reconsolidation (used when Post_Shake=1.0).
p' _{sed,o} [p_sedo]	Default value is -P _{atm} /5. It is the mean effective stress up to which reconsolidation strains are enhanced when Post_Shake=1.0.
c _{rhg} [MC_ratio]	Nominal plastic shear strength ratio used to compute c _{hg} at the time of initialization or when FirstCall is set equal to 0. Default value is 0.005.
c _{hg} [MC_c]	Nominal plastic shear strength assigned at initialization or when FirstCall is set equal to 0. It is computed as the greater of: (1) c _{rhg} times p, and (2) the user-specified value for c _{hg} . Thus, the user-specified value for c _{hg} is the minimum value it will be assigned.

Tracking variables

Many of the parameters internal to PM4Sand3D may be tracked for debugging purposes. These internal parameters are visible through the FLAC3D interface and can be phonetically mapped to different terms in constitutive equations in this manual. Recall that internal parameters (properties) can be accessed using the `zone.prop` command of FLAC3D.

4.2 Example calibration and model responses for a range of element loading conditions

The response of the model is illustrated in this section by presenting simulation results for a set of input parameters that were calibrated to emphasize realistic modeling of liquefaction behavior. Results are presented for sands having initial apparent relative densities of 35%, 55%, and 75% with corresponding SPT (N₁)₆₀ values of approximately 6, 14, and 26, respectively, based on the correlations presented previously. This calibration was performed for a vertical consolidation stress of 100 kPa with a coefficient of lateral earth pressure at rest of 0.5. All secondary input parameters were assigned the default values summarized previously in Table 4.2. Values for G₀ were obtained using the previously presented correlation between SPT (N₁)₆₀ values and overburden-corrected shear wave velocity V_{S1} (Figure 4.1). Values for h_{po} were obtained by matching the CRR values from direct simple shear (DSS) simulations with the CRR_{M=7.5} values that were computed using the SPT-based liquefaction triggering correlation by Idriss and Boulanger (2006, 2008); an SPT-based estimate of CRR for an M=7.5 earthquake and effective overburden stress of 1 atm was assumed to be approximately equal to the CRR

corresponding to 15 uniform loading cycles causing a peak shear strain of 3% in direct simple shear loading. The model input parameters for the examples presented in this section are summarized in Table 4.4 below.

Table 4.4. Input parameters for example element responses

Scenario field condition				Model input parameters ^(a)		
D_R	$(N_1)_{60}$	V_{S1} using Andrus & Stokoe (2000)	$CRR_{M=7.5}$ using Idriss & Boulanger (2008)	D_R	G_o	h_{po}
0.35	5.6	145	0.090	0.35	504	0.22
0.55	14	171	0.147	0.55	717	0.20
0.75	26	196	0.312	0.75	943	0.15

^(a) All secondary input parameters were assigned the default values listed in Table 4.2.

Undrained cyclic loading in direct simple shear (Figures 4.2 – 4.15)

The undrained cyclic loading responses for the calibrated models are illustrated in Figure 4.2 through Figure 4.4 for direct simple shear (DSS). These figures show the stress-strain and stress-path responses for undrained uniform cyclic loading in DSS with a vertical consolidation stress of 1 atm and initial static shear stress ratios (α) of 0.0, 0.1, and 0.2. Results for initial D_R of 35%, 55%, and 75% are presented in Figure 4.2, Figure 4.3, and Figure 4.4, respectively. Close up views of the stress-strain responses for $D_R = 35\%$, 55%, and 75% with an initial static shear stress ratio $\alpha = 0.0$ are presented in Figure 4.5.

The stress-strain responses for $\alpha = 0.0$ illustrate the model's ability to progressively reach larger shear strains with continued cyclic loading, rather than locking up in a repeating loop as many plasticity models do (Ziotopoulou et al. 2014). The ability to simulate the progressive accumulation of shear strains reflects the inclusion of the cumulative fabric terms, as described previously. The progressive increases in peak shear strain after the soil has reached a peak excess pore pressure ratio (r_u) greater than 98% are realistic in magnitude.

The stress-strain responses with nonzero initial static shear stresses show a progressive accumulation of shear strains in the direction of the initial static shear stress, with the rate and nature of the stress-strain response also being reasonably realistic.

CRR versus number of loading cycles – Effect of D_R and failure criterion

The CRR obtained for the calibrated models are summarized in Figure 4.6 showing the cyclic stress ratio (CSR) required to cause excess pore pressure ratio of 98%, and single-amplitude shear strains of 1% and 3% versus number of uniform loading cycles. These results are for DSS loading with a vertical consolidation stress of 1 atm, an initial K_o of 0.5, and zero initial static shear stress ratio ($\alpha=0.0$). The simulation results in this figure were fitted with a power law, for which the exponent " b " is labeled beside each curve.

The slopes of these CRR versus number of loading cycles are in good agreement with typical values obtained in laboratory testing studies. The exponent b is generally between 0.23 and 0.29 for these simulations. For the experimental data in Figure 1.4, the exponent b ranges from a low of 0.1 for one study to a high of 0.34 for another. The ability of the model to produce reasonable slopes for these curves is attributed primarily to the changes in the plastic modulus and dilatancy relationships (Ziotopoulou and Boulanger 2012).

The slopes of the CRR versus number of loading cycle curves depend on the range in the number of loading cycles considered, with the slopes generally being steeper for denser sands at fewer numbers of loading cycles to failure (with their correspondingly greater CSR values). The increase in slope for denser sands at low numbers of loading cycles is related to the sand's greater dilatancy. These slopes can be slightly adjusted by the parameters n^b and n^d . For example, repeating the same simulations with $n^b=0.8$ (versus the value of 0.5 used herein) and with all other factors the same, increases the exponent b to values of 0.27 to 0.47 depending on the range of loading cycles considered. Note that a greater value for n^b also results in greater peak friction angles and changes other responses as well, so such adjustments cannot be made independent of other features of behavior.

CRR versus number of loading cycles – Effect of confining stress

The effect of overburden stress on CRR for the calibrated models is illustrated in Figure 4.7 showing the CSR required to cause a single-amplitude shear strain of 3% versus number of uniform loading cycles for different confining stresses. These results are for DSS loading with initial $K_o=0.5$, initial static shear stress ratio (α) of 0.0, and vertical consolidation stresses of 1, 4, and 8 atm. The cyclic strengths for $D_R=35\%$ are the least affected by confining stress, while the cyclic strengths for $D_R=75\%$ are the most affected (reduced).

The equivalent K_σ values from these simulations, with the CRR values compared at 15 uniform loading cycles, are compared in Figure 4.8 to the relationships recommended by Boulanger and Idriss (2004) based on the framework presented in Boulanger (2003b) (Figure 1.5). The simulated effects of confining stress are in good agreement for looser sands and deviate for denser sands. Where the effects of overburden stresses are important, the user may consider making the calibration dependent on confinement.

CRR versus number of loading cycles – Effect of sustained shear stress

Summary plots of the CSR required to cause a single-amplitude shear strain of 3% versus number of uniform loading cycles are presented in Figure 4.9 for different values of initial static shear stress ratio. Results are presented for sand at $D_R=35\%$, 55%, and 75% loaded in DSS with an initial $K_o=0.5$, a vertical consolidation stress of 1 atm, and with initial static shear stress ratios (α) of 0.0, 0.1, 0.2, and 0.3. The model was initialized at the stress conditions corresponding to the end of 1D vertical consolidation, thereby setting the back-stress ratio terms prior to application of the initial horizontal static shear stress. The simulation results are reasonable in predicting that the presence of an initial static shear stress ratio results in lower cyclic strengths for loose sands (e.g., the $D_R=35\%$ results) and greater cyclic strengths for denser sands (e.g., the $D_R=75\%$ results).

The effect of vertical effective stress on these behaviors is illustrated in Figure 4.10 showing the cyclic resistance ratio (3% peak shear strain in 15 uniform cycles of DSS loading) against initial static

shear stress ratio for D_R of 35, 55, and 75% with vertical effective consolidation stresses of 100 and 400 kPa. Increasing the vertical effective consolidation stress makes the material relatively more contractive, such that the effect of an initial static shear stress ratio is relatively more detrimental at 400 kPa than at 100 kPa. This general pattern agrees with experimental observations (Figure 1.6).

CRR versus number of loading cycles – Effect of K_o

Summary plots of the CSR required to cause a single-amplitude shear strain of 3% versus number of uniform loading cycles are presented in Figure 4.11 for different values of the lateral earth pressure coefficient at rest K_o (i.e., the ratio of horizontal to vertical effective stresses at the time of consolidation). Results are presented for sand at $D_R = 35\%$, 55%, and 75% loaded in DSS, a vertical consolidation stress of 1 atm, and with zero initial static shear stress ratio.

CRR versus number of loading cycles – Effect of perpendicular static and cyclic stresses

The sections above describe behaviors in undrained cyclic DSS loading that can be represented without a three-dimensional model. Figure 4.12 through Figure 4.14 show the material's response to undrained cyclic DSS loading in the presence of a sustained static shear stress perpendicular to the applied cyclic load. These figures show the stress-strain and stress-path responses for undrained uniform cyclic loading in DSS with a vertical consolidation stress of 1 atm and initial perpendicular static shear stress ratios (α) of 0.1, 0.2, and 0.3. Results for initial D_R of 35%, 55%, and 75% are presented in Figure 4.12, Figure 4.13, Figure 4.14, respectively.

The general pattern in which shear strains grow in the direction of the static shear stress agrees with experimental observations (Figure 1.18). In addition, experimental results show that the cyclic resistance ratio decreases in bidirectional cyclic loading for loose to medium dense sands (relative densities of 35% and 45%), as summarized in Figure 1.19. Figure 4.15 compares the cyclic resistance ratio (3% peak shear strain in 15 uniform cycles of DSS loading) for D_R of 35%, 45%, 55%, and 75% and static shear stress ratios (α) of 0.0, 0.1, 0.2, and 0.3 under one directional and bidirectional cyclic loading. As seen in this figure, the cyclic resistance of $D_R = 35\%$ and 45% reduces with increasing static bias as also observed in experimental data. For $D_R = 55\%$ the cyclic resistance remains largely unaffected for lower static biases and mildly increases for high static biases, in contrast with experimental data (Figure 1.19). For $D_R = 75\%$ the cyclic resistance increases across all values of static bias but no comparison to experimental data is available.

Undrained triaxial cyclic loading (Figures 4.16 – 4.24)

The undrained cyclic loading responses for the calibrated models are illustrated in Figure 4.16 through Figure 4.18 for triaxial loading (TX). These figures show the stress-strain and stress-path responses for undrained uniform cyclic loading in TX with a vertical consolidation stress of 1 atm and initial static shear stress ratios (α) of 0.0, 0.1, and 0.2. In TX loading an initial static shear stress ratio of 0 implies an isotropic test. Nonzero initial static stress ratios result in anisotropic cyclic tests. The static shear stress ratio in triaxial loading was calculated from the shear and normal stresses on the potential failure planes following Idriss and Boulanger 2008 and Seed et al. 1975. Results for initial D_R of 35%, 55%, and 75% are presented in Figure 4.16, Figure 4.17, and Figure 4.18, respectively.

Close-up views of the stress-strain responses for $D_R = 35\%$, 55% , and 75% for the isotropic cases are presented in Figure 4.19. These stress-strain responses illustrate the model's ability to progressively reach larger axial strains with continued cyclic loading. In addition, the asymmetrical strain accumulation between triaxial extension and triaxial compression seen experimentally is reflected.

CRR versus number of loading cycles – Effect of D_R and failure criterion

The TX CSR was defined as one half times the maximum cyclic deviatoric stress normalized by the isotropic consolidation stress. The CRR obtained for the calibrated models are summarized in Figure 4.20 showing the CSR required to cause single-amplitude axial strains of 1%, 3%, and 7.5% versus number of uniform loading cycles. These results are for isotropic TX loading with a vertical consolidation stress of 1 atm. The simulation results in this figure were fitted with a power law, for which the exponent "b" is labeled beside each curve. As in DSS, the range of exponents fits within the experimental range.

CRR versus number of loading cycles – Effect of confining stress

The effect of overburden stress on CRR for the calibrated models is illustrated in Figure 4.21 showing the CSR required to cause a single-amplitude axial strain of 3% versus number of uniform loading cycles for different confining stresses. These results are for isotropic TX loading under vertical consolidation stresses of 1, 4, and 8 atm. The cyclic strengths for $D_R = 35\%$ are the least affected by confining stress, while the cyclic strengths for $D_R = 75\%$ are the most affected (reduced). Note that the effect of confining stress is of smaller magnitude than in DSS loading.

The equivalent K_σ values from these simulations, with the CRR values compared at 15 uniform loading cycles, are compared in Figure 4.22 to the relationships recommended by Boulanger and Idriss (2004) based on the framework presented in Boulanger (2003b) (Figure 1.5). The simulated effects of confining stress in triaxial conditions differ from those shown previously for DSS loading.

CRR versus number of loading cycles – Effect of sustained shear stress

Summary plots of the CSR required to cause a single-amplitude axial strain of 3% versus number of uniform loading cycles are presented in Figure 4.23 for different values of initial static shear stress ratio. Results are presented for sand at $D_R = 35\%$, 55% , and 75% loaded in TX under isotropic consolidation with a $K_o = 1.0$ under a consolidation stress of 1 atm, then loaded to initial static shear stress ratios (α) of 0.0, 0.1, 0.2, and 0.3. The simulation results in this show that initial anisotropy in triaxial loading has less impact on the cyclic strengths with varying density than was shown in DSS cyclic loading. However, minor increases in cyclic strength exist for denser sands (e.g., the $D_R = 75\%$ results).

The effect of vertical effective stress on these behaviors is illustrated in Figure 4.24 showing the cyclic resistance ratio (3% peak axial strain in 15 uniform cycles of TX loading) against initial static shear stress ratio for D_R of 35, 55, and 75% with vertical effective consolidation stresses of 100 and 400 kPa.). As mentioned above, the impact of static shear stress and consolidation stress has only minor impacts on CRR in TX loading. The effects of static shear stress on cyclic strength of denser sands underestimates the increases observed in lab experiments (Figure 1.6).

Drained monotonic loading in direct simple shear, plane strain- and triaxial-compression (Figures 4.25 – 4.28)

The response for drained monotonic loading in direct simple shear (DSS), plane strain compression (PSC), and triaxial compression (TXC) for sand at D_R of 35%, 55%, and 75% under vertical confining stresses of $\frac{1}{4}$, 1, 4, and 16 atm is shown in Figures 4.25, 4.26, and 4.27 respectively. The responses reasonably approximate the effects of relative density and confining stress on both the stress-strain and volumetric strain responses. The plots show responses up to shear strains of 10%, while the simulations tend to reach critical state conditions at shear strains of 40-70%. The post-peak rate of strain-softening is dictated by the dilation rate, which is constrained to approximate Bolton's (1986) stress-dilatancy relationship. The simulated post-peak softening is slower than often observed in experimental results (e.g., Figure 1.9) because drained laboratory experiments are often affected by strain localizations in dilating sands (e.g., Desrues et al. 1996, Sadrekarimi and Olson 2010); the rate of strain-softening in a dilating zone is much lower than represented by global measurements of stress and strain.

The peak effective friction angles from simulations of drained monotonic loading in TXC, PSC and DSS are shown versus vertical consolidation stress in Figure 4.28 where they are also compared to Bolton's (1986) relationship for $Q=10$ and $R=1.5$ (which are the default values of Q and R that have been selected for the model). The peak friction angles in DSS are lower than predicted by Bolton's relationship because the dilation rate is based on assumptions of triaxial loading, as described in Section 2.7. In addition, the friction angles for TXC and DSS are computed differently. For TXC, the peak friction angle is computed based on the peak stress ratio within the element, without any predetermined assumptions regarding the orientation of the plane on which it will occur. For DSS loading, the peak friction angle was computed as the inverse tangent of the peak stress ratio on the horizontal plane, following the same convention commonly used in practice for interpreting such tests. In the DSS simulation, however, the horizontal plane was not the plane of maximum stress obliquity, and therefore the interpreted peak friction angle is slightly lower than the value obtained in TXC. Computationally, both the DSS and TXC mobilize similar peak friction angles if the comparison is made only for the plane of maximum stress obliquity in both simulations; in the DSS, however, the stress ratio on the horizontal plane in the DSS simulations is often closer to $\sin(\phi)$ as opposed to $\tan(\phi)$, which results in the apparent differences shown in Figure 4.28. The peak friction angles for plane strain compression are greater than Bolton's relationship at higher stresses because of the assumed Lode angle dependency, as discussed previously in Section 2.3. Despite these differences, the peak friction angles are reasonable and consistent with typical design correlations (e.g., Kulhawy and Mayne 1990).

Undrained monotonic loading in direct simple shear and triaxial compression (Figures 4.29– 4.32)

The undrained monotonic loading in direct simple shear (DSS) and in triaxial compression for sand at D_R of 35%, 55%, and 75% under vertical consolidation stresses of $\frac{1}{4}$, 1, 4, and 16 atm are shown in Figure 4.29 and Figure 4.31, while the same responses are shown with normalization by the vertical consolidation stress in Figure 4.30 and Figure 4.32. The stress-strain responses show strain hardening behavior at lower relative states than would be expected based on some experimental results for reconstituted sands, such as presented by Yoshimine et al. (1999), or some case history-based residual strength correlations. The calibration of the model to specific values for critical state or residual undrained shear strength is discussed in Section 4.3.

Drained, strain-controlled, cyclic direct simple shear from small to large strains

Drained strain-controlled cyclic loading in DSS for sand at D_R of 35%, 55%, and 75% under vertical consolidation stresses of 1, 4, and 16 atm with $K_o=1.0$ are shown in Figure 4.33 through Figure 4.35, with results also shown for the equivalent modulus reduction (G/G_{max}) and equivalent damping ratio (ξ) versus cyclic shear strain amplitude (γ). Also shown on these figures are the modulus reduction and equivalent damping ratio curves recommended for sands at different depths by EPRI (1993). The simulated modulus reduction and equivalent damping ratio curves depend on the effective confining stress in a pattern and magnitude that is consistent with empirical design correlations, such as the ones by EPRI. The simulated modulus reduction curves for this calibration tend to be slightly higher than the empirical curves, whereas the simulated equivalent damping ratios are in reasonable agreement with the empirical curves over a fairly broad range of shear strain amplitudes. The model response in DSS for this calibration avoids the problem common to many plasticity models of producing excessively high equivalent damping ratios as shear strain amplitudes approach about one percent.

Drained, strain-controlled cyclic direct simple shear and triaxial loading: Densification under large numbers of cycles

Drained strain-controlled cyclic loading in DSS and TXC for sand at D_R of 35%, 55%, and 75% subjected to 20 cycles at 1% shear strain under a vertical effective stress of 1 atm are presented in Figure 4.36 and Figure 4.37, respectively, to illustrate the accumulation of volumetric strains with increasing number of constant-amplitude strain cycles. The model response in DSS with this calibration produces volumetric strains that are about twice the values expected based on the empirical data presented in Figure 1.13, although the general pattern of stress-strain behavior and its dependency on D_R and confining stress are reasonably consistent with the empirical data. Alternative model calibrations can produce better agreement with these behaviors, but they were generally found to require compromising the fit to the CRR correlations.

Post-liquefaction reconsolidation strains

The volumetric strains that develop during post-liquefaction reconsolidation of sand are difficult to model using the conventional separation of strains into elastic and plastic components because a large portion of the post-liquefaction reconsolidation strains are due to sedimentation effects (i.e., volume reductions while the effective stresses remain close to zero) which are not easily incorporated into either the elastic or plastic components of behavior. For example, it is common for many plasticity-based constitutive models to predict reconsolidation volumetric strains from a condition of $r_u=100\%$ that are only a fraction of one percent (Ziotopoulou and Boulanger 2013b, whereas experimental data show values ranging from one to four percent for most relative densities (e.g., Figure 1.13).

Volumetric strains due to post-cyclic reconsolidation, with and without the PostShake option, are plotted in Figure 4.38 versus the maximum shear strain induced during undrained cyclic loading. Results are shown for sand at $D_R = 35\%$, 55%, and 75% loaded in DSS with an initial $K_o=0.5$, a vertical consolidation stress of 1 atm, and zero initial static shear stress ratio. After cyclic loading to different maximum shear strains, the horizontal shear stress was reduced to zero such that the excess pore pressure was near its maximum possible value (e.g., r_u was approximately 98% or larger for cases with maximum shear strains of 3% or greater). The computed volumetric strains were less than about 0.25% with PostShake = 0 (default value) and are much smaller than expected based on common experimental

data. The computed volumetric strains with PostShake = 1 (imposed at the end of cyclic loading) are in much closer agreement with experimental data (Figure 1.14).

4.3 Example calibration with user-defined critical state undrained shear strength

Calibration of the model to a specified undrained critical state shear strength ($s_{u,cs}$) is illustrated in this section. The $s_{u,cs}$ for any set of calibration parameters can be computed from the critical state line (Figure 2.1) as,

$$s_{u,cs} = \frac{M}{2} \frac{P_A}{100} e^{\left(Q - \frac{R}{D_R}\right)} \quad (1)$$

The critical state line parameters Q and R can be adjusted to fit an experimentally determined critical state line (e.g., Kamai and Boulanger 2012) or fit an $s_{u,cs}$ estimated from a case history-based correlation. Boulanger and Ziotopoulou (2018) illustrated how the model can be calibrated to a specific target $s_{u,cs}$ by setting R according to,

$$R = D_R \left[Q - \ln \left(\frac{2s_{u,cs}}{M} \frac{100}{P_A} \right) \right] \quad (2)$$

The logic of this approach is illustrated in Figure 4.39 wherein the target $s_{u,cs}$ is used to compute p'_{cs} given M , which is then used to compute R given D_R and Q . The dependence of M on Lode Angle can be estimated using figures presented previously. Also, the above expression can be rearranged to obtain the target $s_{u,cs}$ by adjusting Q instead of R .

The calibration of R or Q to obtain a target $s_{u,cs}$ is based on the $s_{u,cs}$ corresponding to the current D_R . If a numerical simulation procedure imposes undrained conditions (no fluid flow) such that the D_R does not change significantly during the simulation, then the model will produce the target $s_{u,cs}$ at large shear strains. If a numerical simulation allows fluid flow, then the D_R may increase or decrease throughout the simulation depending on the pore pressure diffusion pattern. In this latter case, the resulting changes in $s_{u,cs}$ during the simulation (per the above equations) need to be recognized by the user.

Consider the example calibration for $D_R = 35\%$ in Section 4.2. This example calibration, referred to as Calibration 1 below, used $G_o = 504$ and $h_{po} = 0.22$ with all other parameters retaining their default values. Recall that Q and R have default values of 10 and 1.5, respectively. For DSS loading, M may be estimated as approximately 1.1 times the value for triaxial compression loading, arriving an estimated value of 1.42. This calibration should therefore produce a $s_{u,cs}$ value of,

$$s_{u,cs} = \frac{1.42}{2} \frac{101.3 \text{ kPa}}{100} e^{\left(10 - \frac{1.5}{0.35}\right)} = 218 \text{ kPa}$$

For a zone with $\sigma'_{vc} = 100$ kPa and $K_o = 0.5$, this corresponds to an initially dense-of-critical-state condition that produces $s_{u,cs}/\sigma'_{vc} = 2.18$. The simulated response for monotonic undrained DSS loading, as shown in Figure 4.40, exhibits the expected strain hardening response.

A revised calibration, referred to as Calibration 2, is now presented for a case where the calibration includes targeting an $s_{u,cs}/\sigma'_{vc} = 0.07$ for a zone with $\sigma'_{vc} = 100$ kPa and $K_o = 0.5$. The target $s_{u,cs}$ is therefore only 7.1 kPa, which is achieved by setting R to,

$$R = 0.35 \left[10 - \ln \left(\frac{2(7.1 \text{ kPa})}{1.42} \frac{100}{101.3 \text{ kPa}} \right) \right] = 2.698$$

The target V_{s1} and $CRR_{M=7.5}$ values remain unchanged, which means G_o remains unchanged whereas h_{po} had to be recalibrated to 0.80.

The responses of the above two calibrations to monotonic undrained DSS loading with $\sigma'_{vc} = 100$ kPa are shown in Figure 4.40. Calibration 1 results in a strongly dilative response whereas Calibration 2 produces the intended contractive response.

The responses of the above two calibrations to cyclic undrained DSS loading with $\sigma'_{vc} = 100$ kPa and $K_o = 0.5$, and an imposed $CSR = 0.090$ are shown in Figure 4.41. Calibration 1 produces a cyclic mobility response wherein cyclic shear strains progressively accumulate during cyclic loading. The specimen never develops flow liquefaction because it is always dense of critical state. Calibration 2 exhibits flow liquefaction, which is triggered by the generation of excess pore pressures during the cyclic loading. The specimen develops flow liquefaction because the peak shear stresses imposed during cyclic loading are greater than the $s_{u,cs}$.

The representation of post-liquefaction residual shear strengths in practice is complicated by various challenges and limitations as discussed in Boulanger et al. (2012, 2015) and Boulanger and Ziotopoulou (2018). The $s_{u,cs}$ mobilized in the field, and the timing at which it is mobilized, will depend on pore pressure diffusion and associated void redistribution processes, which depend on the soil properties (e.g., initial relative density, cyclic strength, permeability), subsurface stratigraphy (e.g., layer thicknesses, slope angles, continuity of interfaces), and ground motion characteristics (e.g., shaking intensity, shaking duration, shaking history). Simulation of void redistribution processes and their timing are highly uncertain, such that pragmatic approximations for representing post-triggering residual shear strengths during and after strong shaking are often used in practice (see Boulanger et al. 2015 for additional discussion).

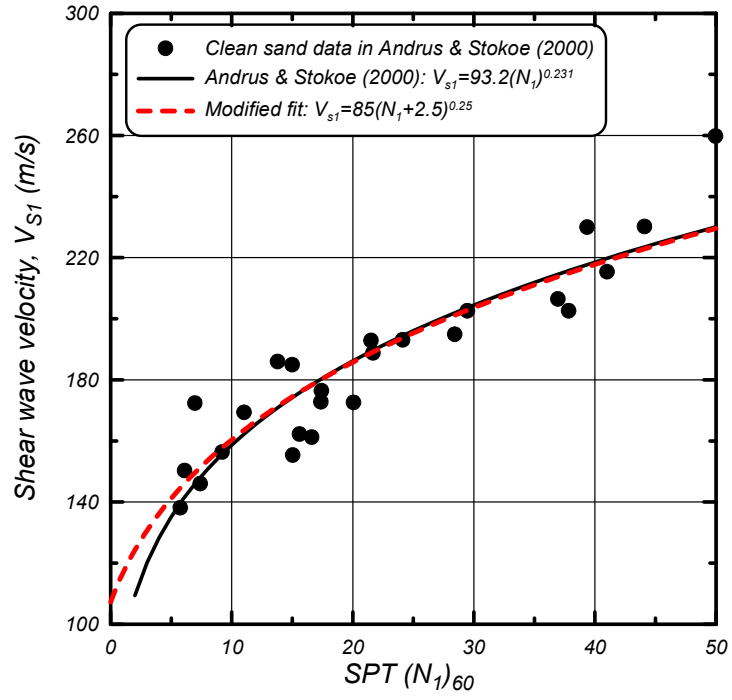


Figure 4.1. Correlation between overburden-corrected shear wave velocity and SPT penetration resistances in clean sands (after Andrus and Stokoe 2000).

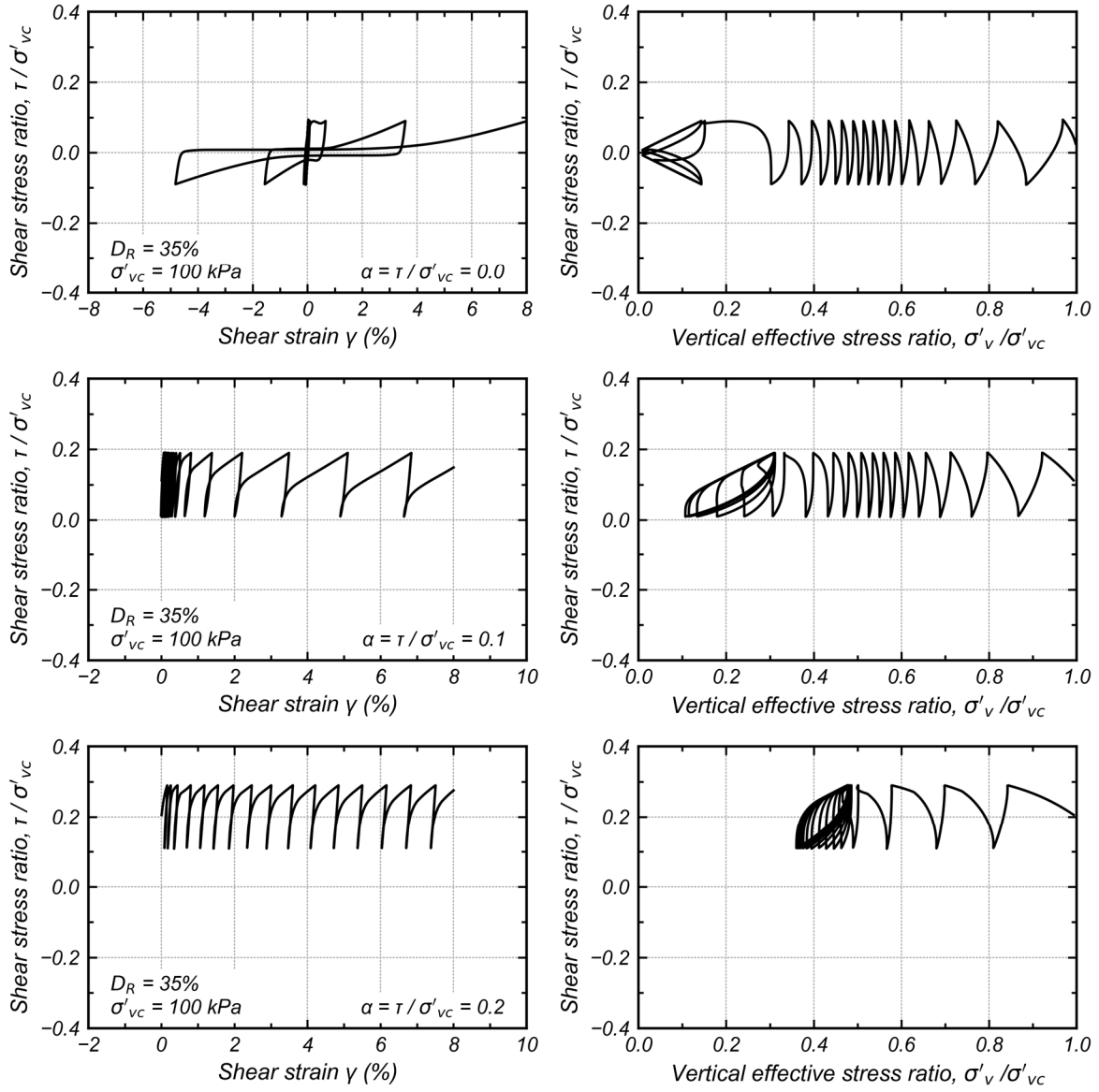


Figure 4.2. Undrained cyclic DSS loading response for $D_R = 35\%$ with vertical effective consolidation stress of 1 atm and with initial static shear stress ratios of 0.0, 0.1, and 0.2.

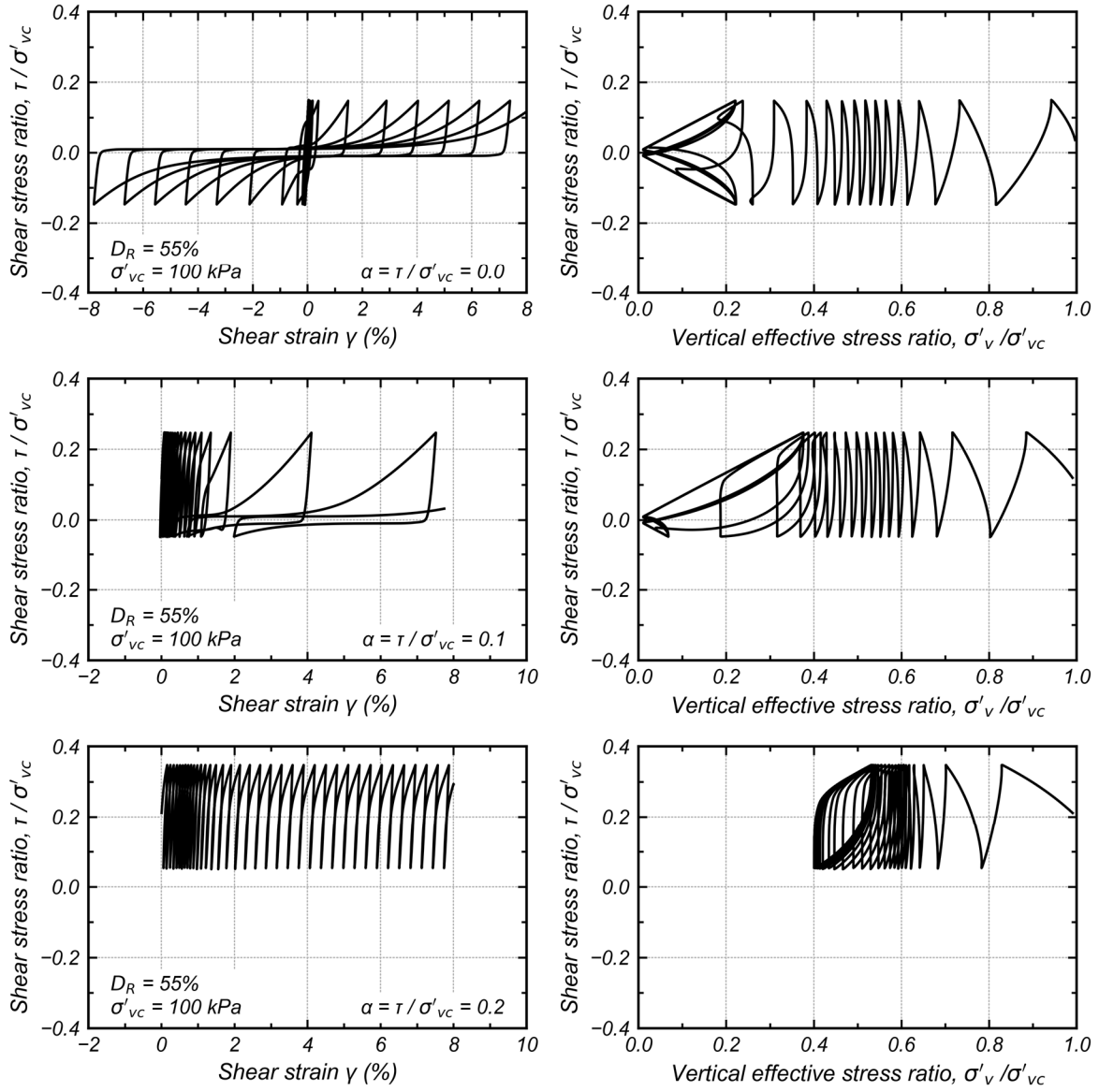


Figure 4.3. Undrained cyclic DSS loading response for $D_R = 55\%$ with vertical effective consolidation stress of 1 atm and with initial static shear stress ratios of 0.0, 0.1, and 0.2.

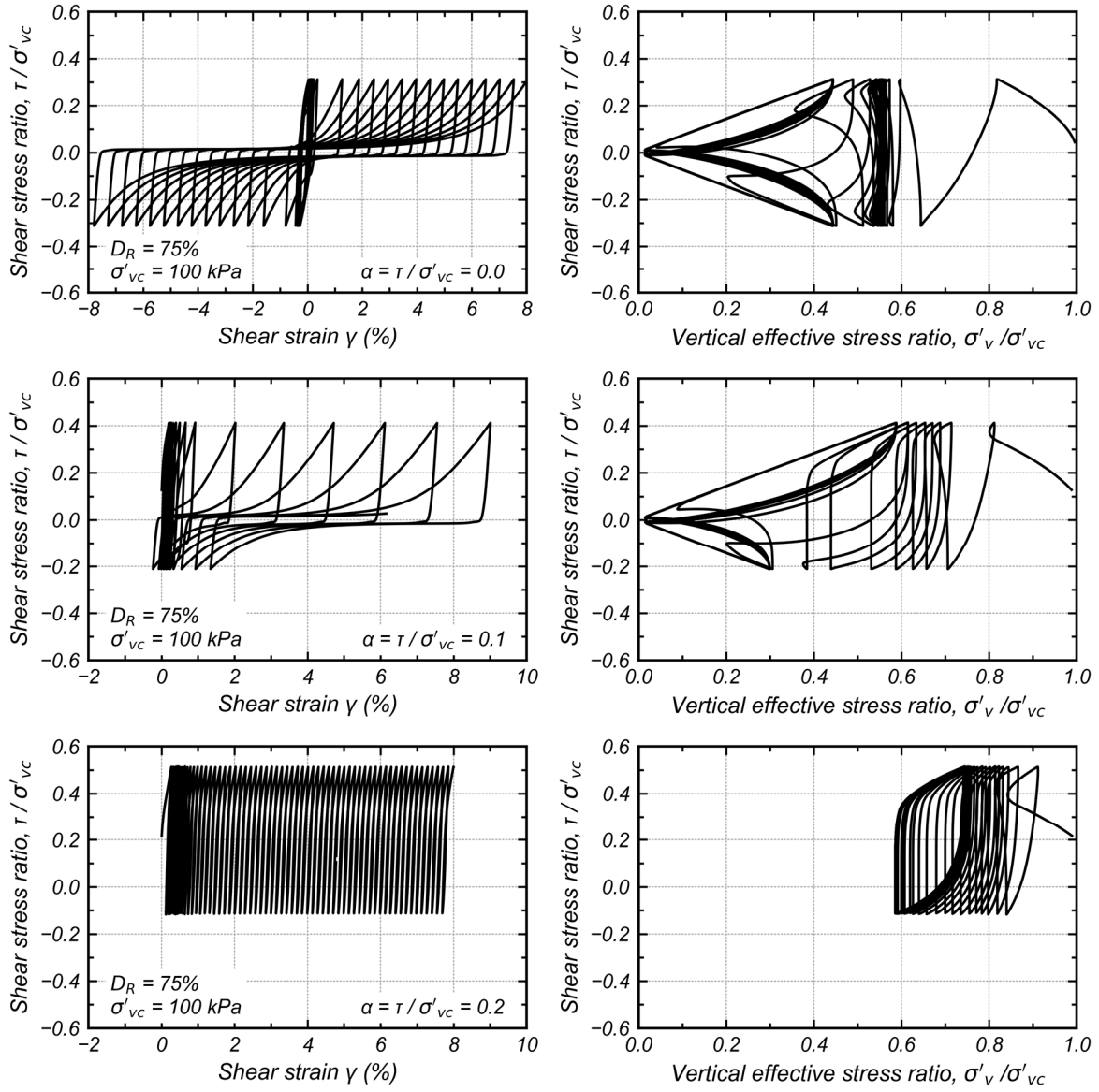


Figure 4.4. Undrained cyclic DSS loading response for $D_R = 75\%$ with vertical effective consolidation stress of 1 atm and with initial static shear stress ratios of 0.0, 0.1, and 0.2.

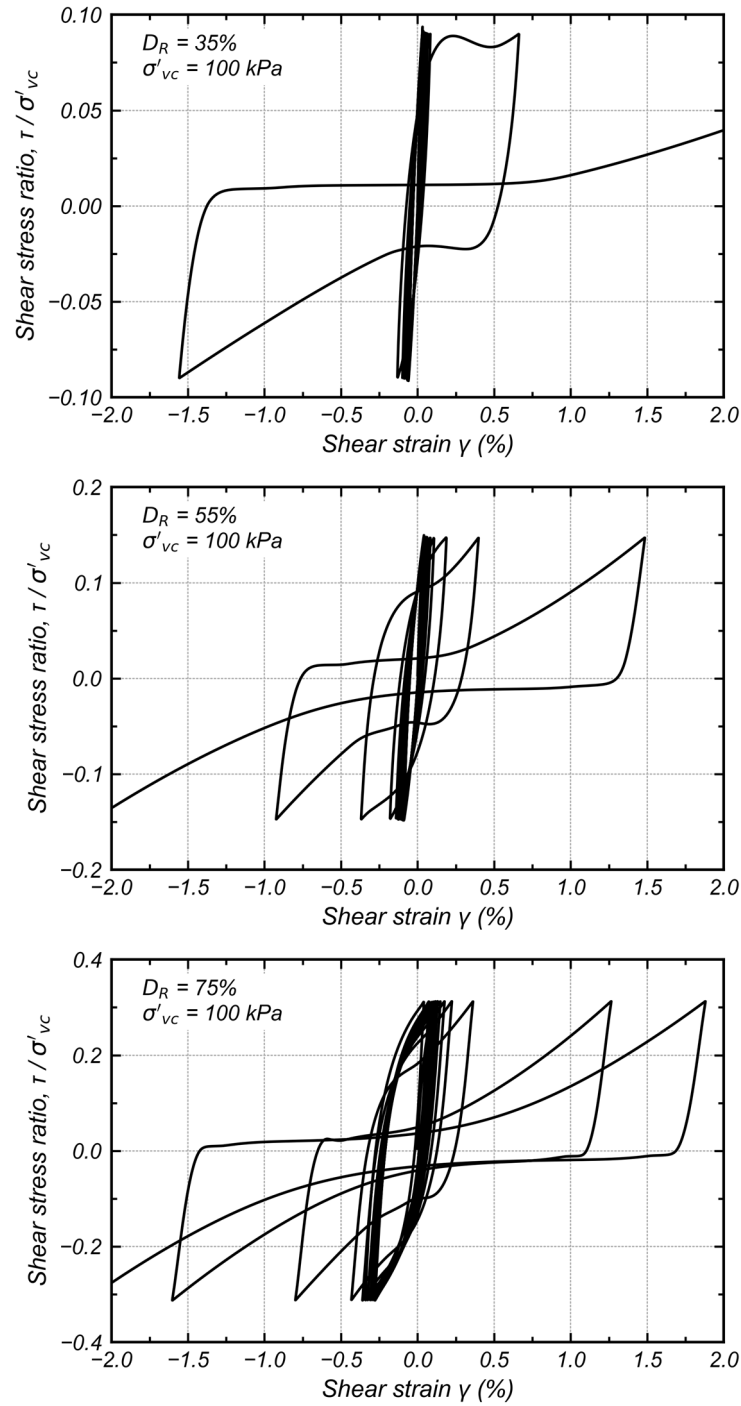


Figure 4.5. Undrained cyclic DSS loading responses for $D_R = 35, 55$, and 75% with a vertical effective consolidation stress of 1 atm and without any initial static shear stress.

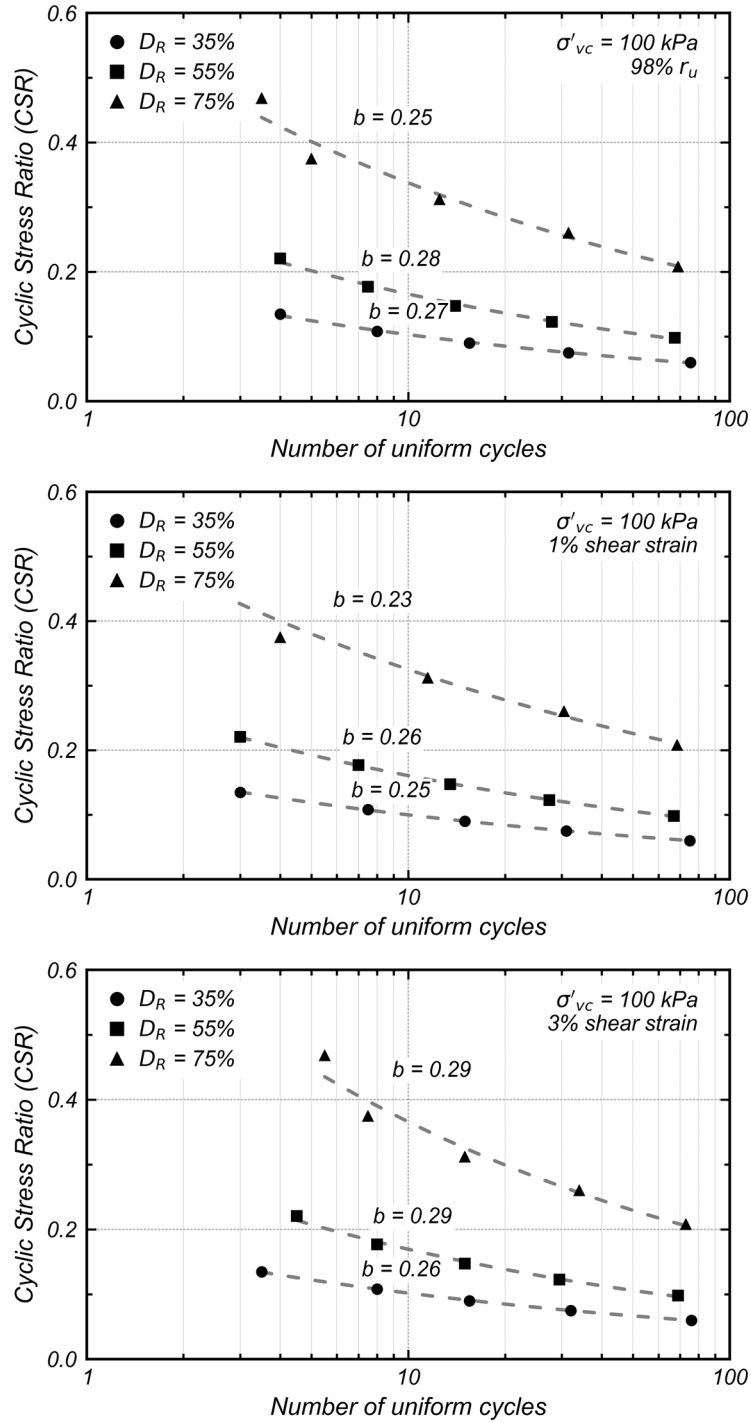


Figure 4.6. Cyclic stress ratios versus number of equivalent uniform loading cycles in undrained DSS loading to cause single-amplitude shear strains of 1%, 3%, or 7.5% for $D_R = 35, 55$, and 75% with a vertical effective consolidation stress of 1 atm. Each set of CSR-N simulations was fit with a power relationship and the exponent b labeled beside each curve.

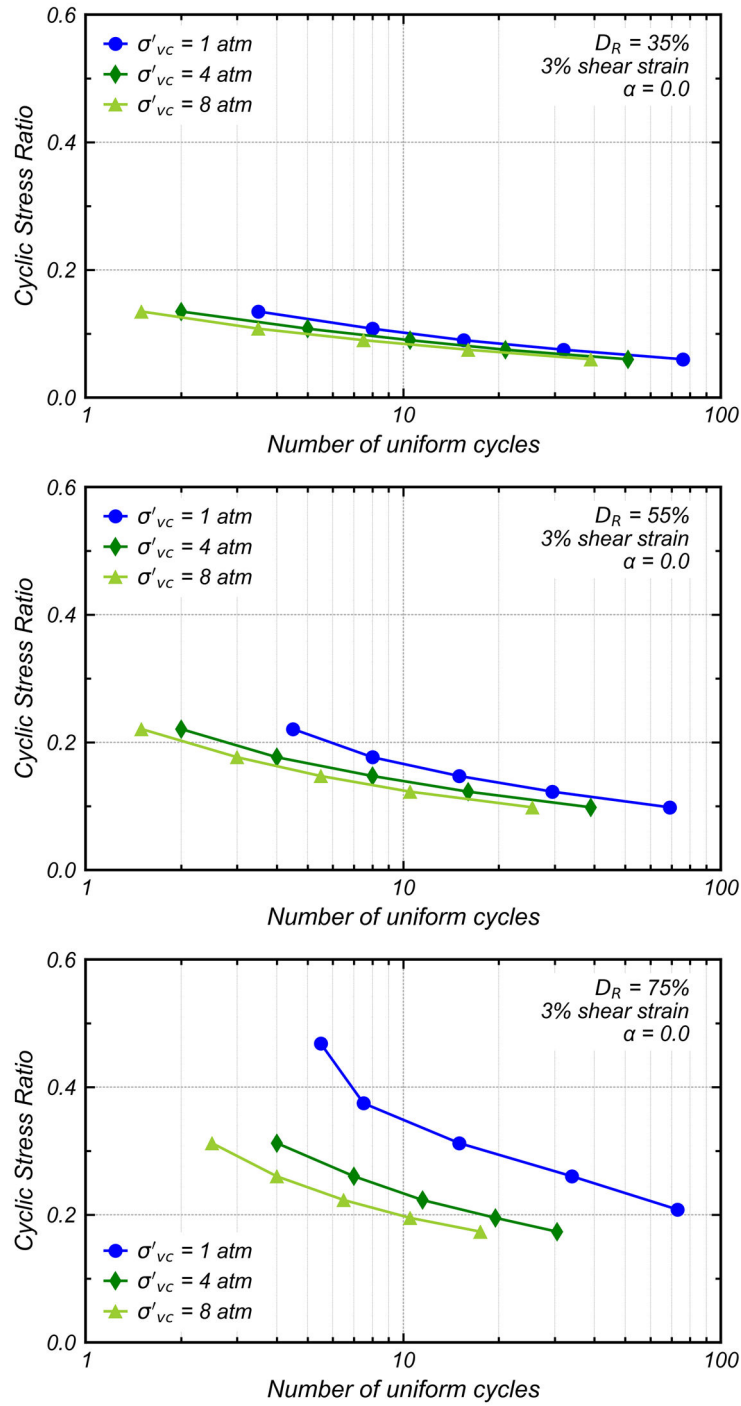


Figure 4.7. Cyclic stress ratios versus number of equivalent uniform loading cycles in undrained DSS loading to cause single-amplitude shear strain of 3% for $D_R = 35, 55$, and 75% with vertical effective consolidation stresses of 1, 4, and 8 atm.

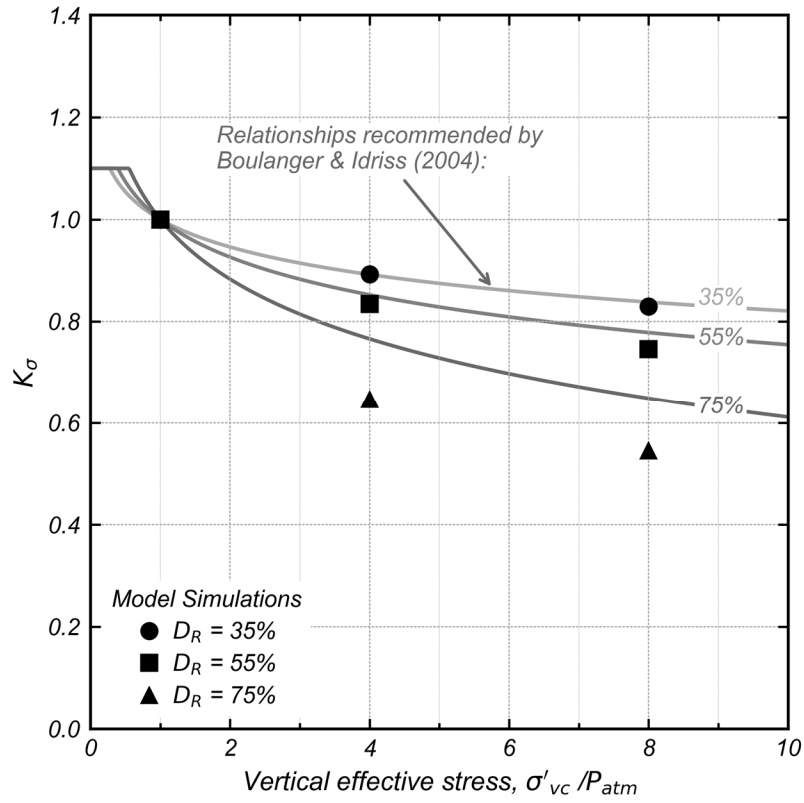


Figure 4.8. Comparison of K_σ factors, determined at 15 uniform loading cycles to cause 3% single-amplitude shear strain, from DSS simulations versus relationships recommended by Boulanger and Idriss (2004).

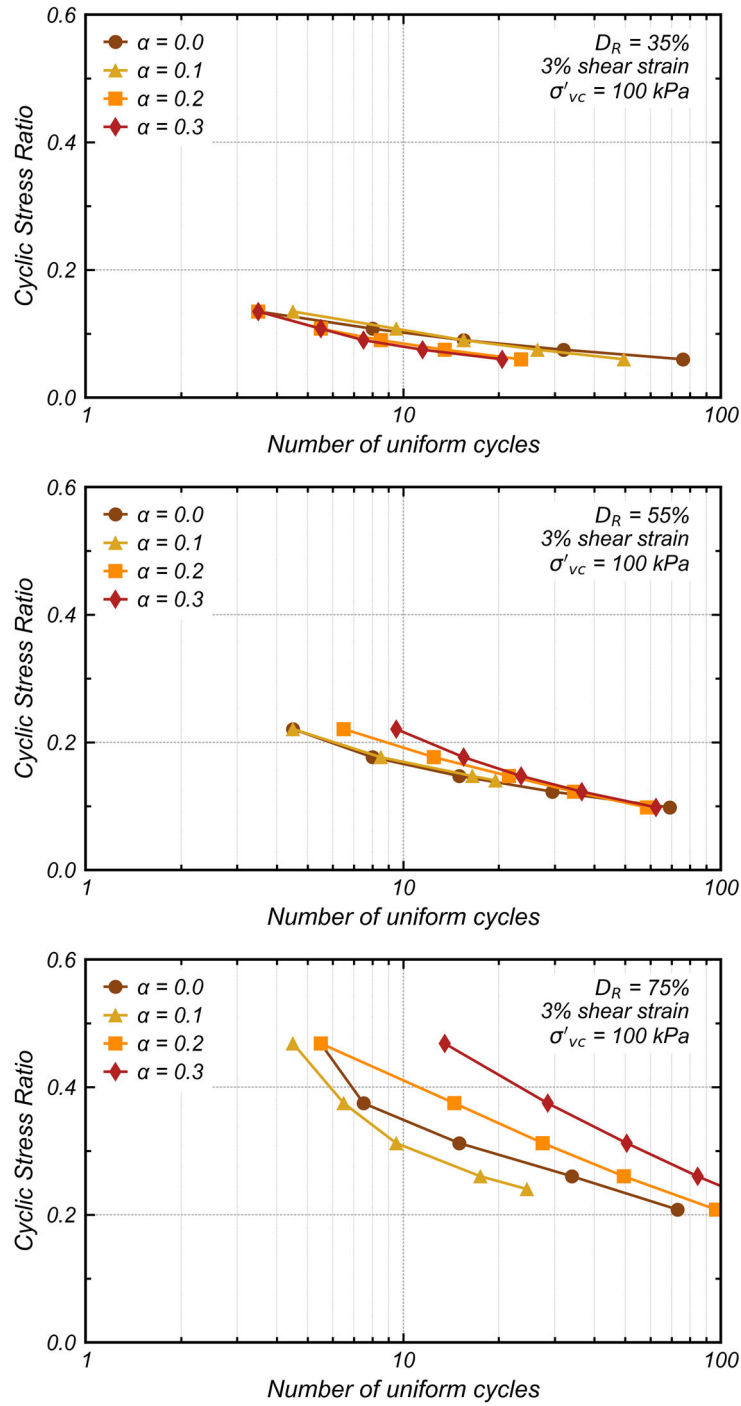


Figure 4.9. Cyclic stress ratios versus number of equivalent uniform loading cycles in undrained DSS loading to cause single-amplitude shear strain of 3% for $D_R = 35, 55$, and 75% with vertical effective consolidation stresses of 1 atm and initial static shear stress ratios of 0.0, 0.1, 0.2, and 0.3.

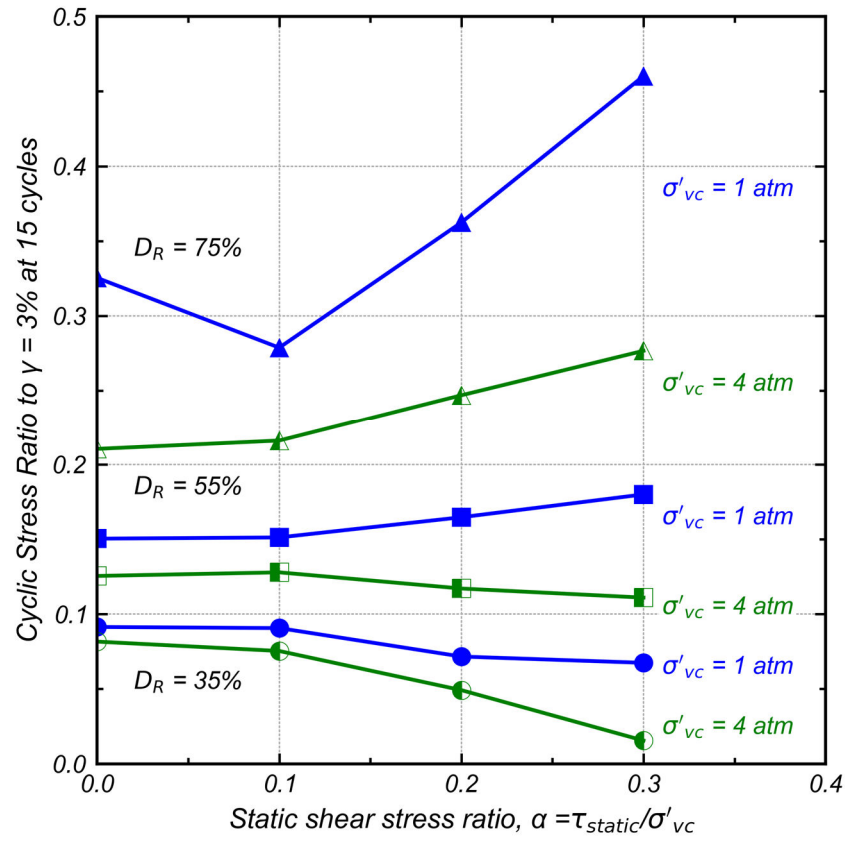


Figure 4.10. Effect of vertical effective consolidation stresses of 1 and 4 atm on the variation of cyclic resistance ratio (peak shear strain of 3% in 15 uniform cycles in DSS loading) for sand at D_R of 35, 55, and 75% with initial static shear stress ratios of 0.0, 0.1, 0.2, and 0.3.

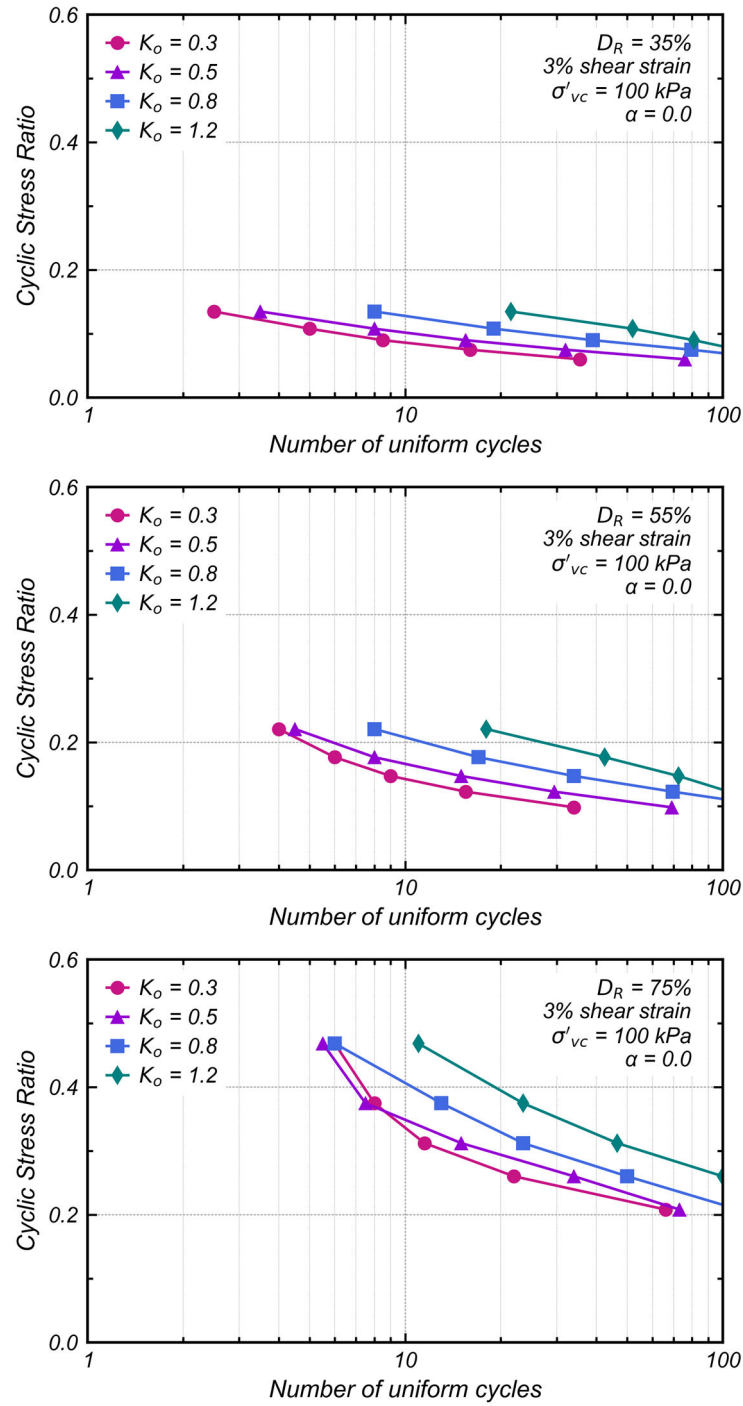


Figure 4.11. Cyclic stress ratios versus number of equivalent uniform loading cycles in undrained DSS loading to cause single-amplitude shear strain of 3% for $D_R = 35, 55$, and 75% with vertical effective consolidation stresses of 1 atm and initial K_o values of 0.3, 0.5, 0.8, and 1.2.

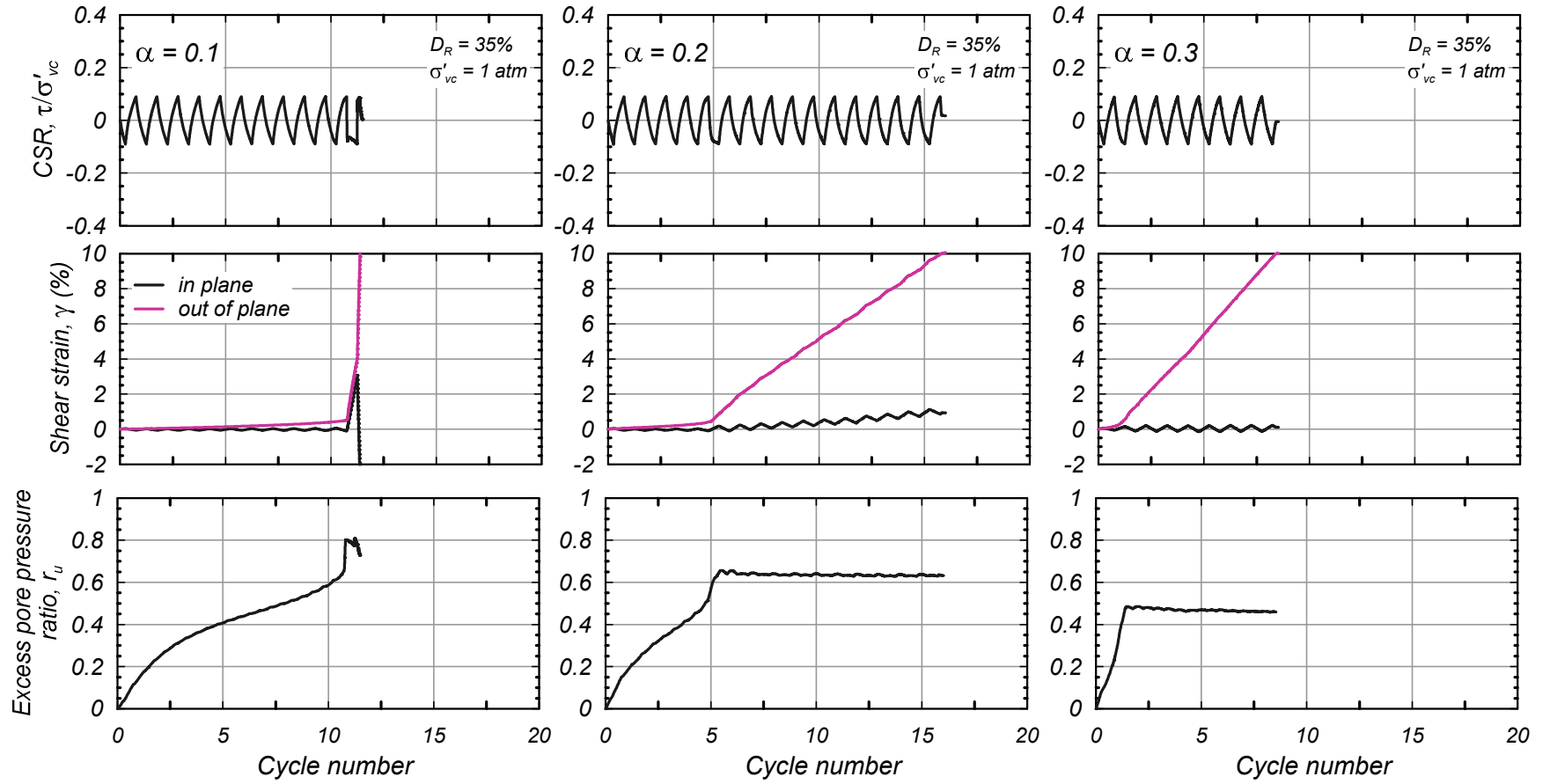


Figure 4.12. Undrained bidirectional cyclic DSS loading response for $D_R = 35\%$ with vertical effective consolidation stress of 1 atm and with applied static shear stress ratios of 0.1, 0.2, and 0.3 perpendicular to the cyclic load.

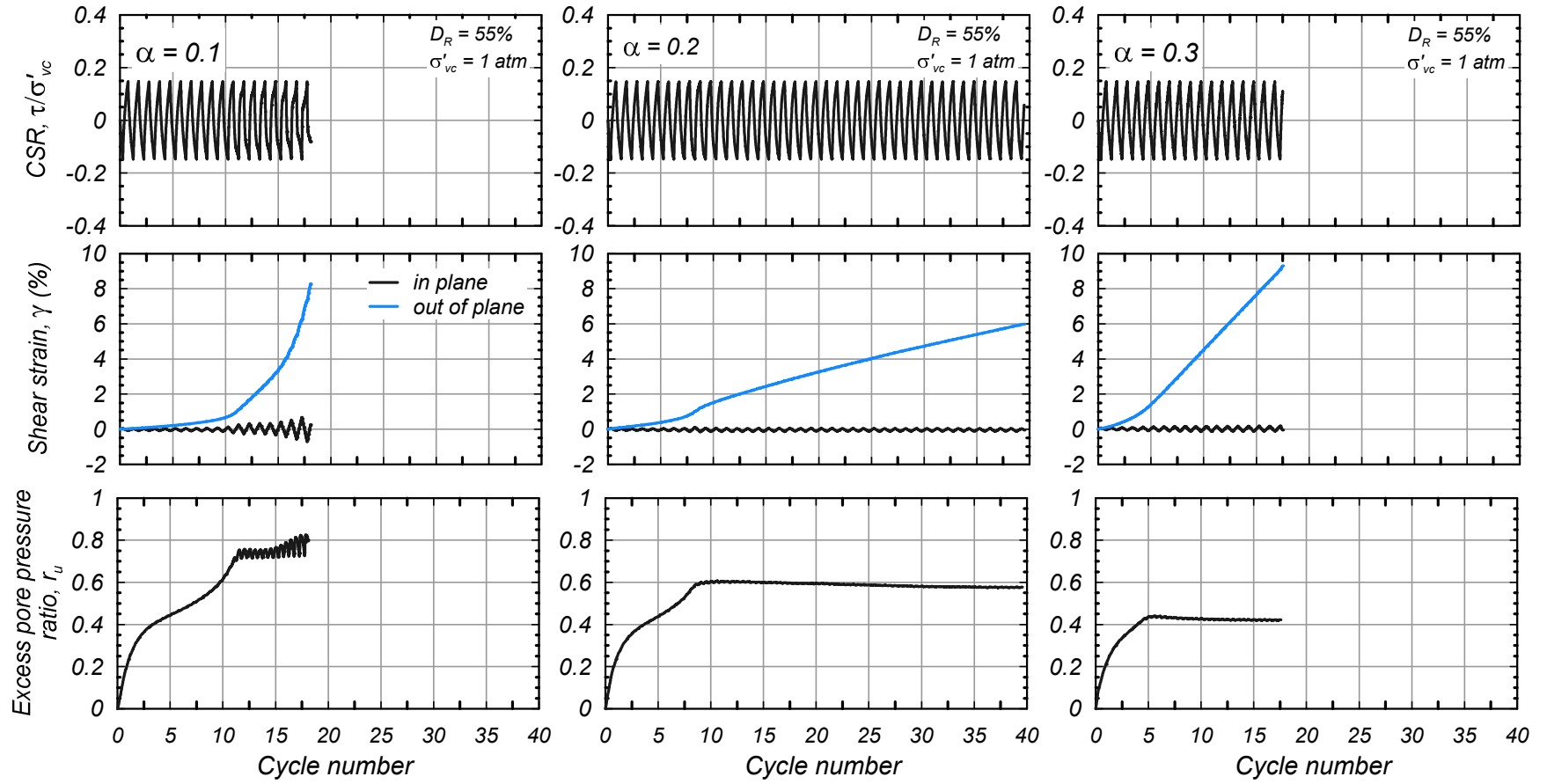


Figure 4.13. Undrained bidirectional cyclic DSS loading response for $D_R = 55\%$ with vertical effective consolidation stress of 1 atm and with applied static shear stress ratios of 0.1, 0.2, and 0.3 perpendicular to the cyclic load.

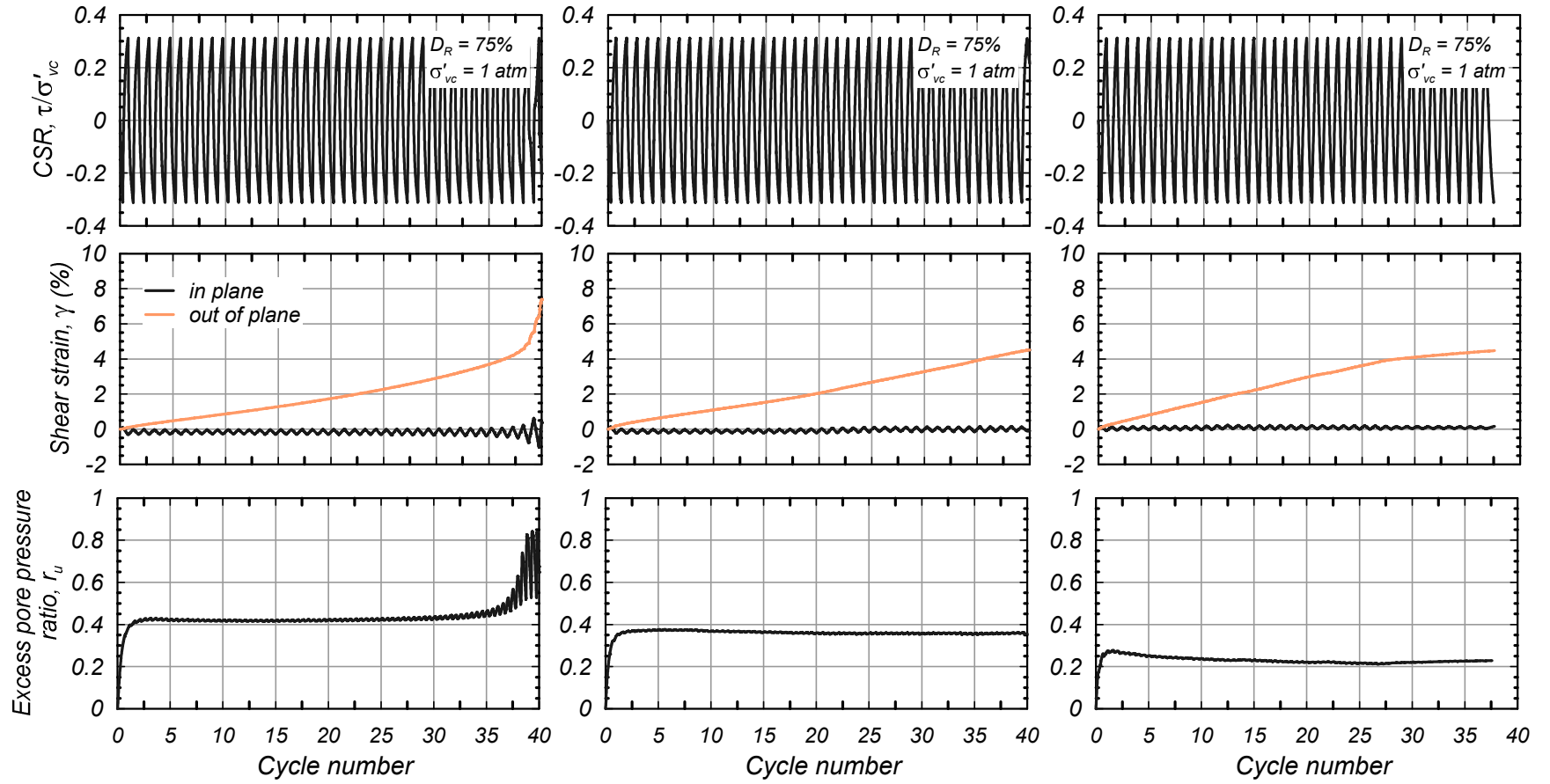


Figure 4.14. Undrained bidirectional cyclic DSS loading response for $D_R = 75\%$ with vertical effective consolidation stress of 1 atm and with applied static shear stress ratios of 0.1, 0.2, and 0.3 perpendicular to the cyclic load.

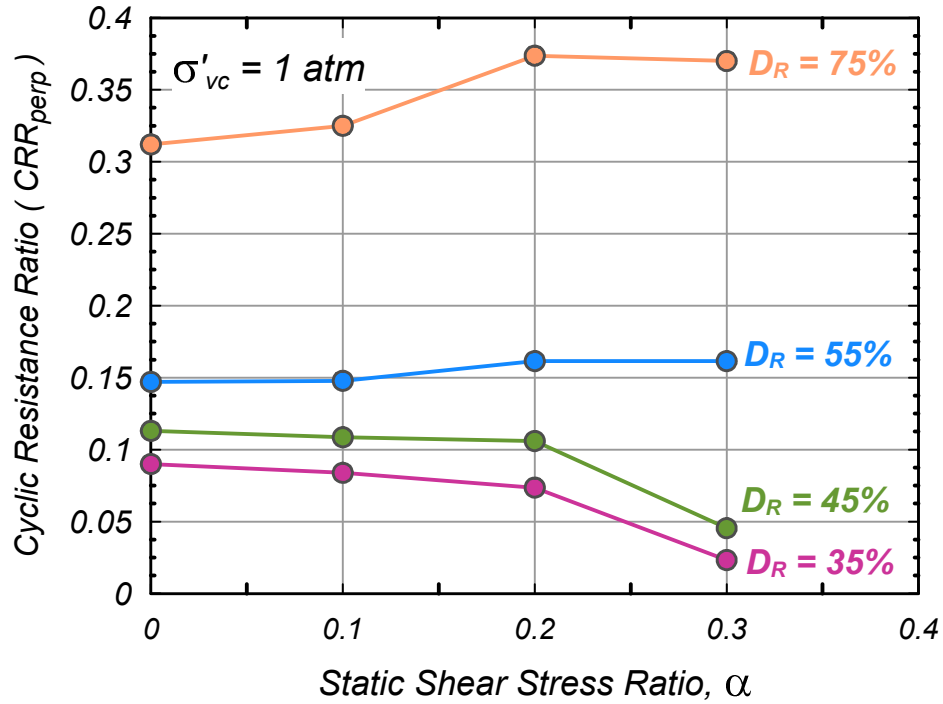


Figure 4.15. Dependence of cyclic resistance (determined at 15 uniform loading cycles to cause 3% single-amplitude shear strain) on initial static shear stress applied perpendicular to the cyclic load in bidirectional direct simple shear tests for relative densities of 35%, 45%, 55%, and 75%, under a consolidation stress of 1 atm.

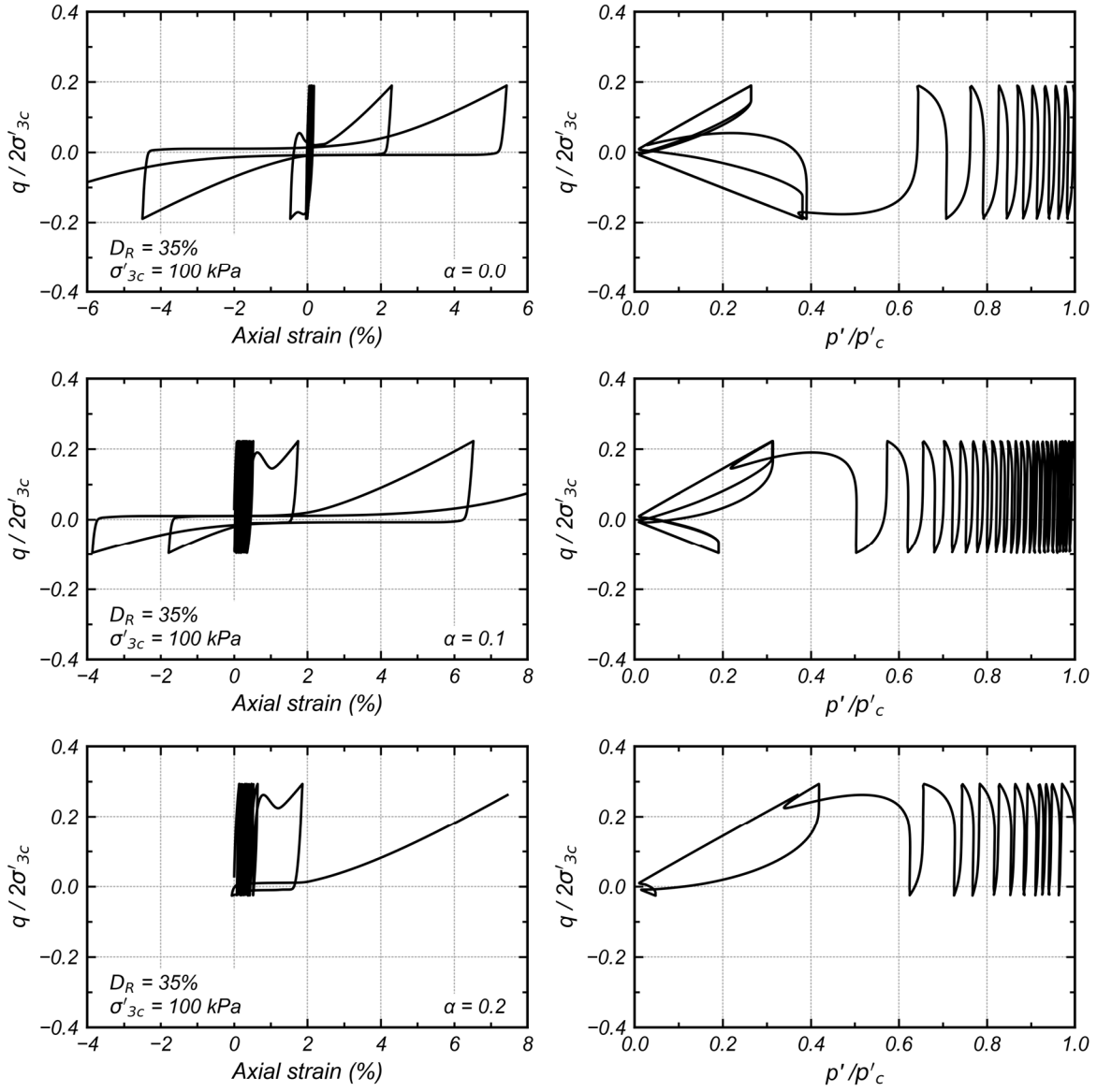


Figure 4.16. Undrained cyclic TXC loading response for $D_R = 35\%$ with vertical effective consolidation stress of 1 atm and with initial static shear stress ratios of 0.0, 0.1, and 0.2.

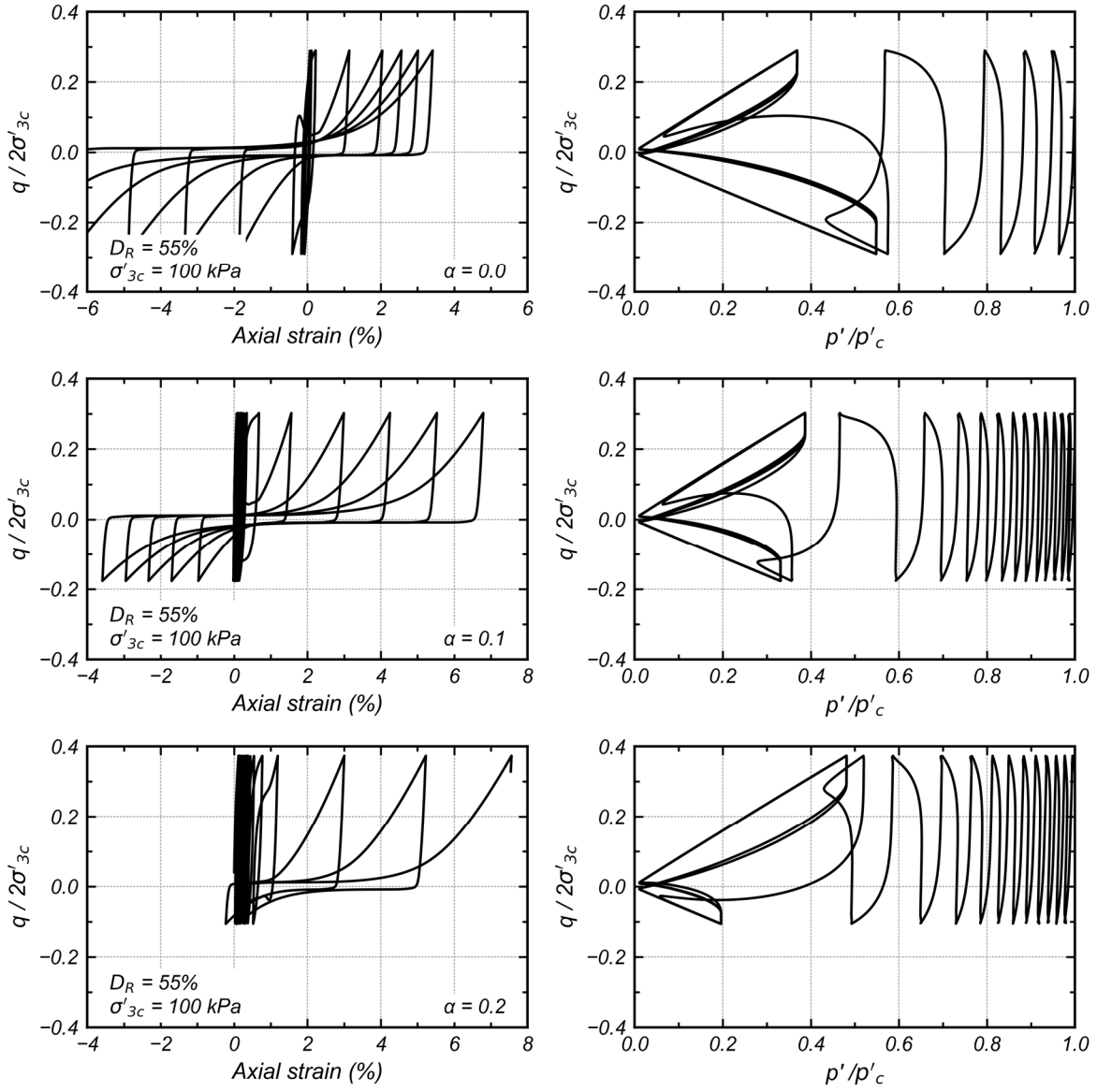


Figure 4.17. Undrained cyclic TXC loading response for $D_R = 55\%$ with vertical effective consolidation stress of 1 atm and with initial static shear stress ratios of 0.0, 0.1, and 0.2.

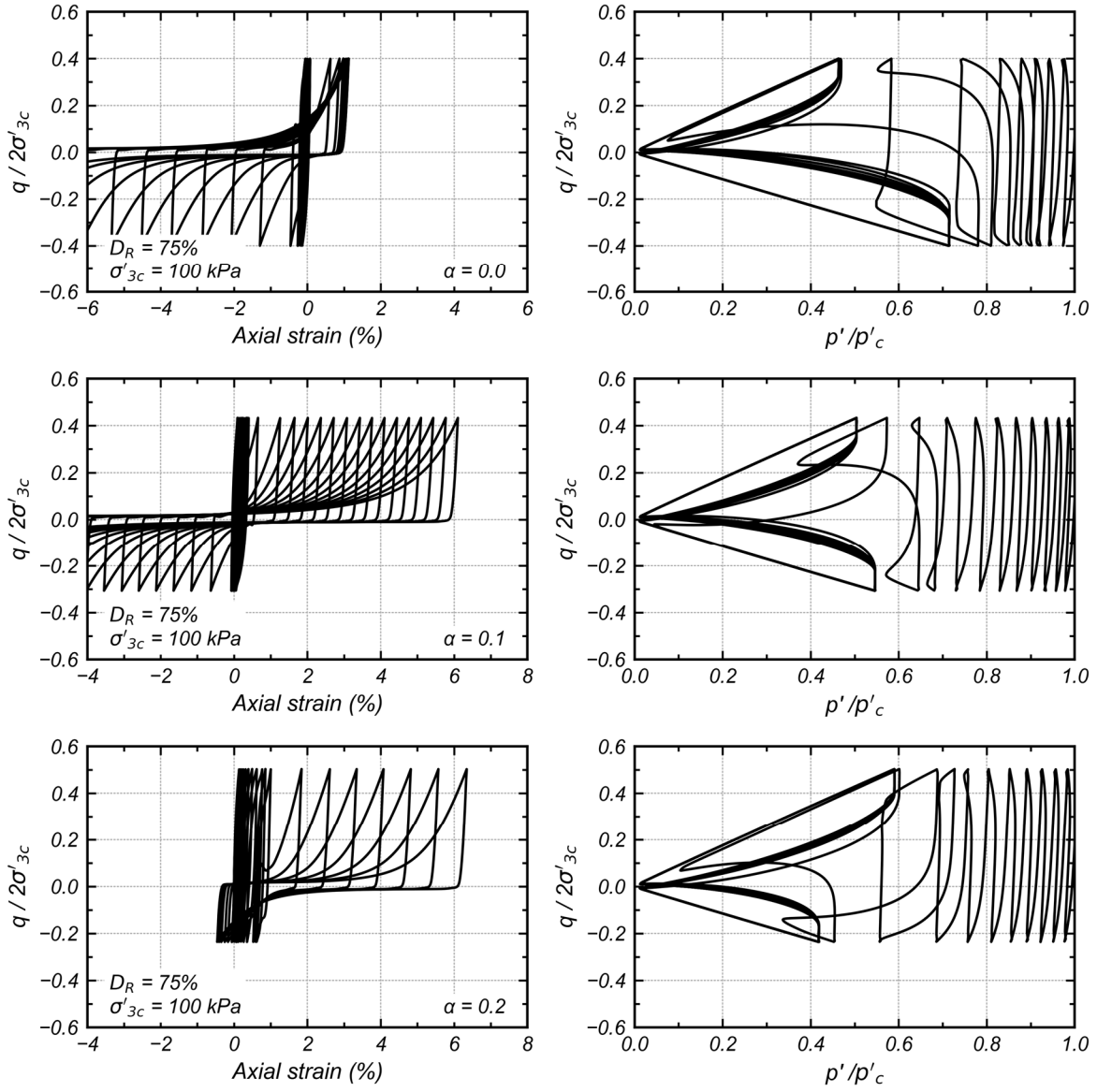


Figure 4.18. Undrained cyclic TXC loading response for $D_R = 75\%$ with vertical effective consolidation stress of 1 atm and with initial static shear stress ratios of 0.0, 0.1, and 0.2.

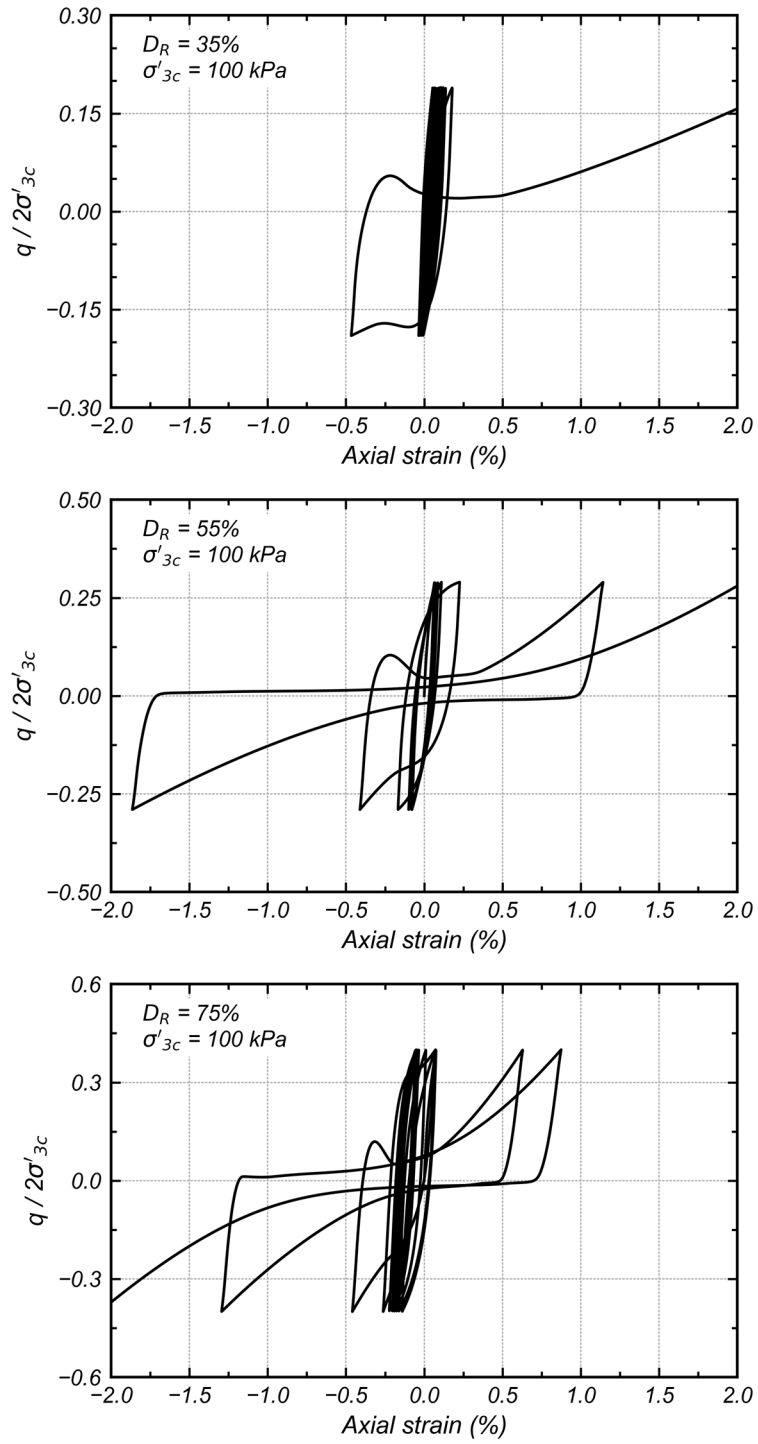


Figure 4.19. Undrained cyclic TXC loading responses for $D_R = 35, 55,$ and 75% with a vertical effective consolidation stress of 1 atm and without any initial static shear stress.

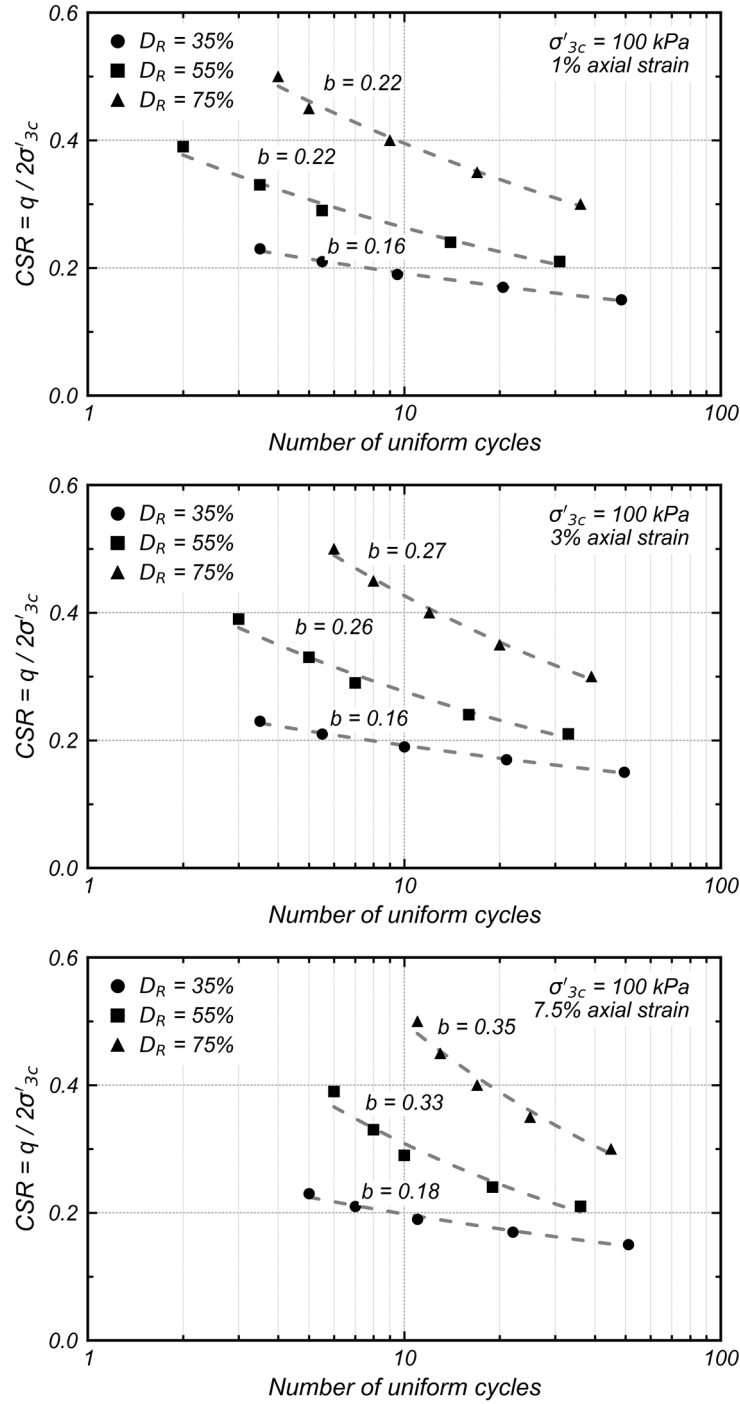


Figure 4.20. Cyclic stress ratios versus number of equivalent uniform loading cycles in undrained TXC loading to cause single-amplitude axial strains of 1%, 3%, or 7.5% for $D_R = 35$, 55, and 75% with a vertical effective consolidation stress of 1 atm. Each set of CSR-N simulations was fit with a power relationship and the exponent b labeled beside each curve.

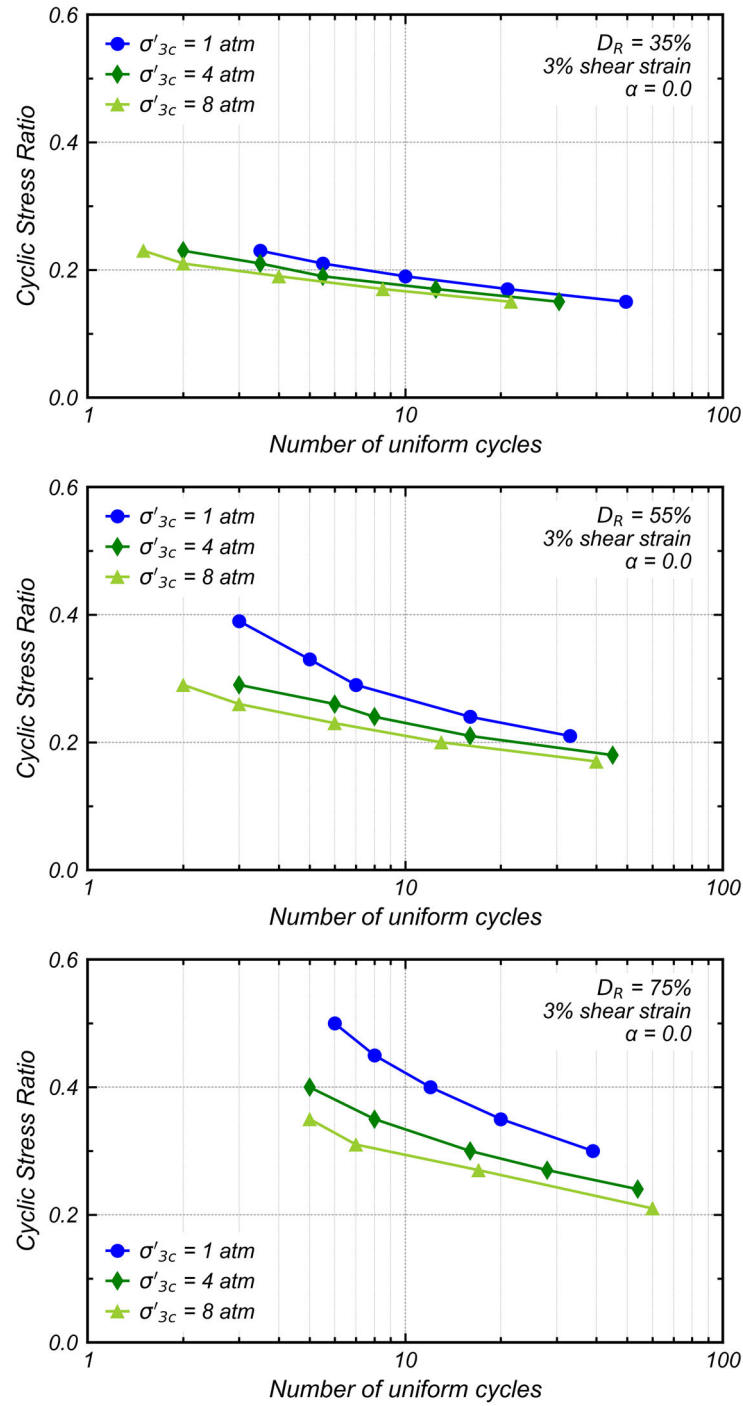


Figure 4.21. Cyclic stress ratios versus number of equivalent uniform loading cycles in undrained TXC loading to cause single-amplitude axial strain of 3% for $D_R = 35, 55$, and 75% with vertical effective consolidation stresses of 1, 4, and 8 atm.

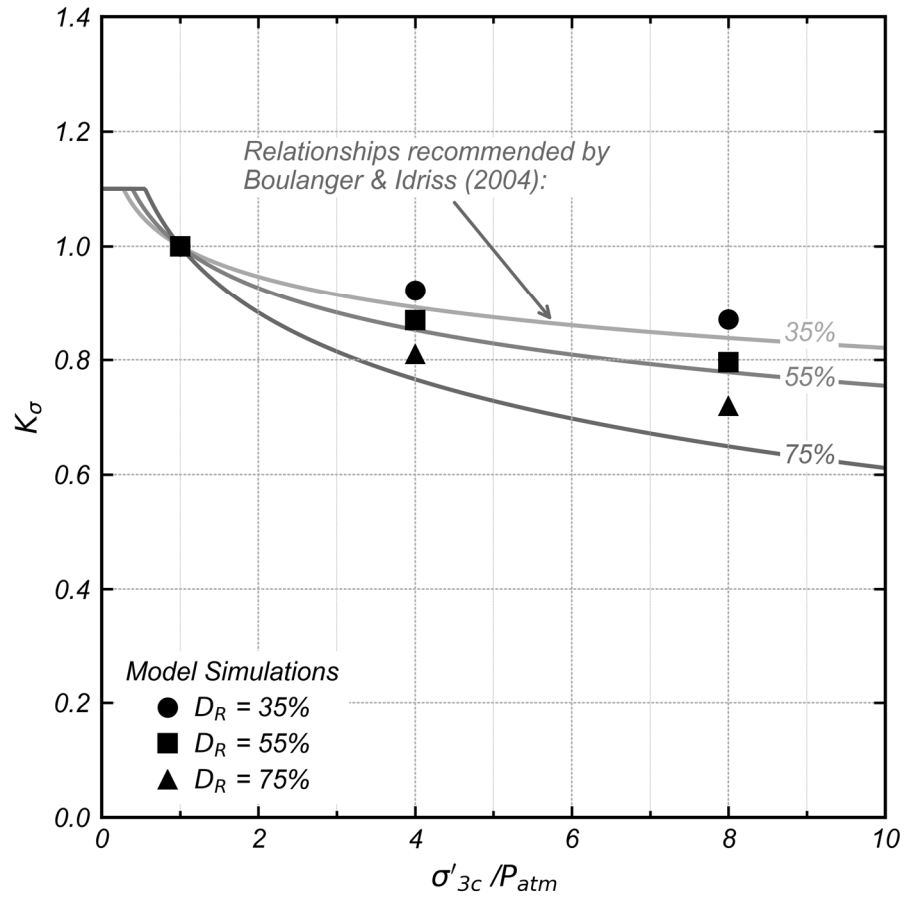


Figure 4.22. Comparison of K_σ factors, determined at 15 uniform loading cycles to cause 3% single-amplitude axial strain, from TXC simulations versus relationships recommended by Boulanger and Idriss (2004).

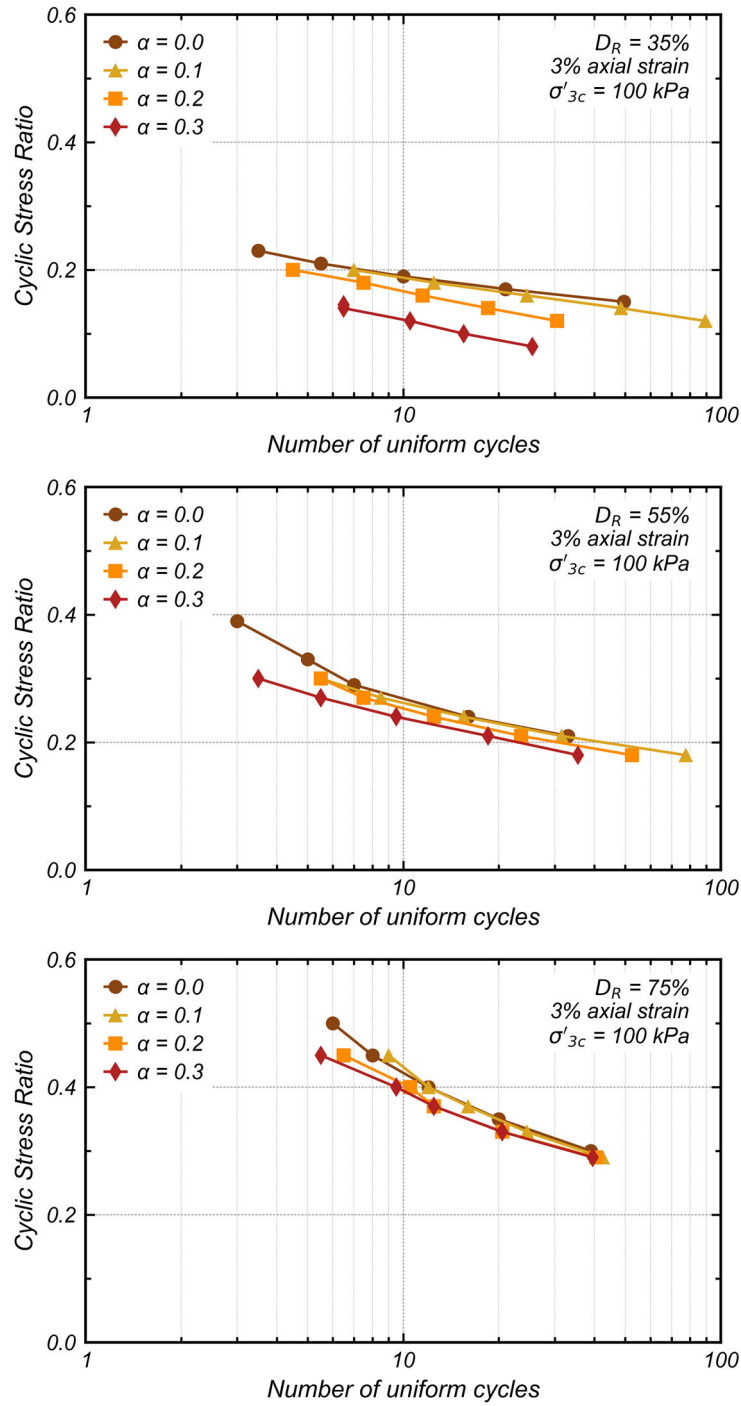


Figure 4.23. Cyclic stress ratios versus number of equivalent uniform loading cycles in undrained TXC loading to cause single-amplitude axial strain of 3% for $D_R = 35, 55$, and 75% with vertical effective consolidation stresses of 1 atm and initial static shear stress ratios of 0.0, 0.1, 0.2, and 0.3.

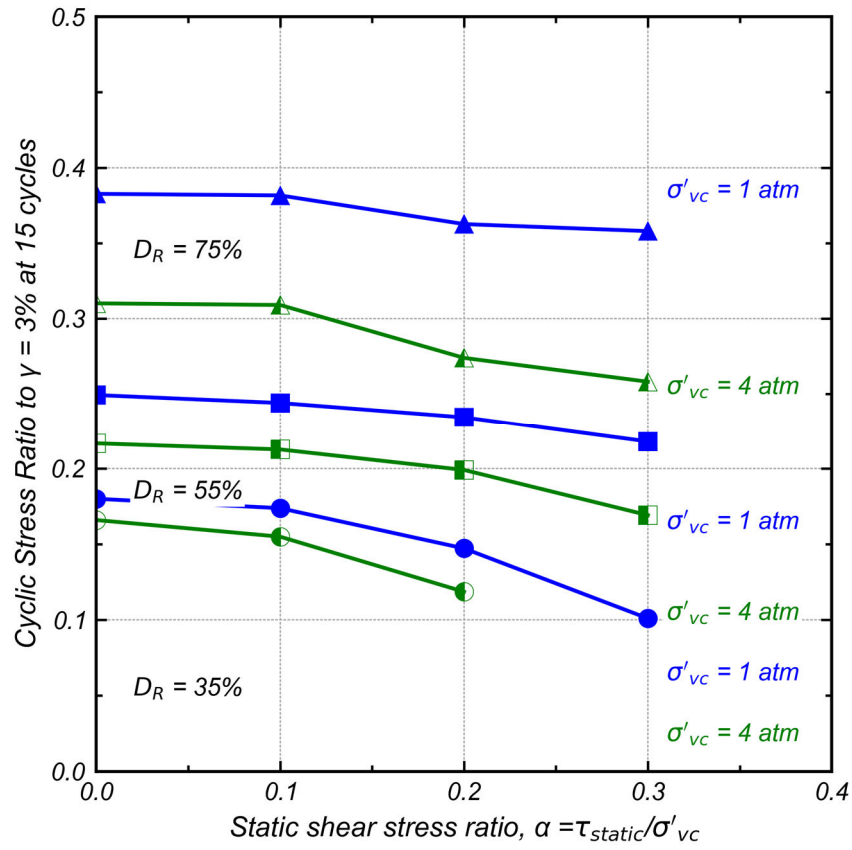


Figure 4.24. Effect of vertical effective consolidation stresses of 1 and 4 atm on the variation of cyclic resistance ratio (peak shear strain of 3% in 15 uniform cycles in TXC loading) for sand at D_R of 35, 55, and 75% with initial static shear stress ratios of 0.0, 0.1, 0.2, and 0.3.

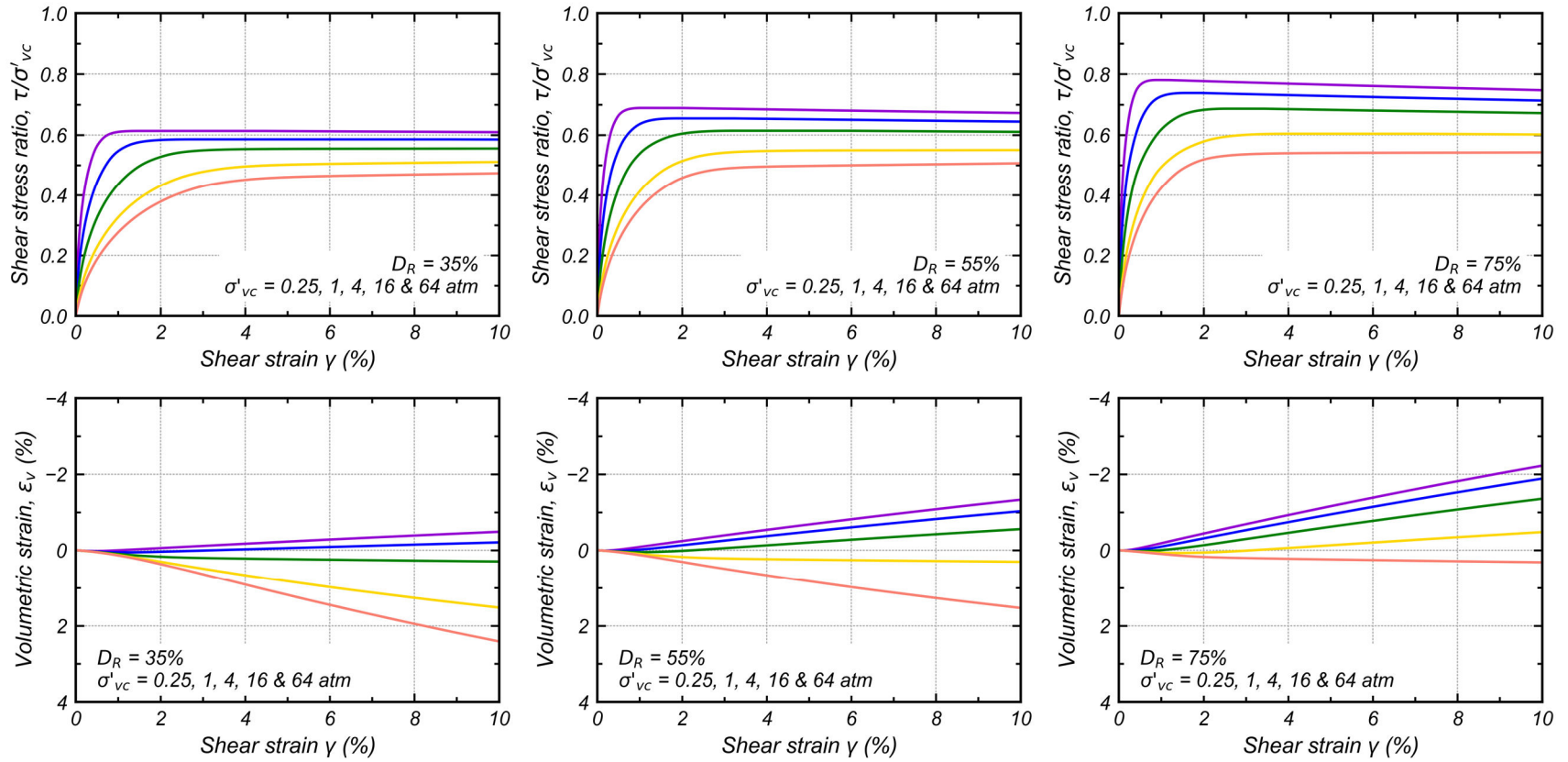


Figure 4.25. Drained monotonic DSS loading responses for $D_R = 35, 55$, and 75% with vertical effective confining stresses of $1/4, 1, 4, 16$, and 64 atm and $K_0=0.5$.

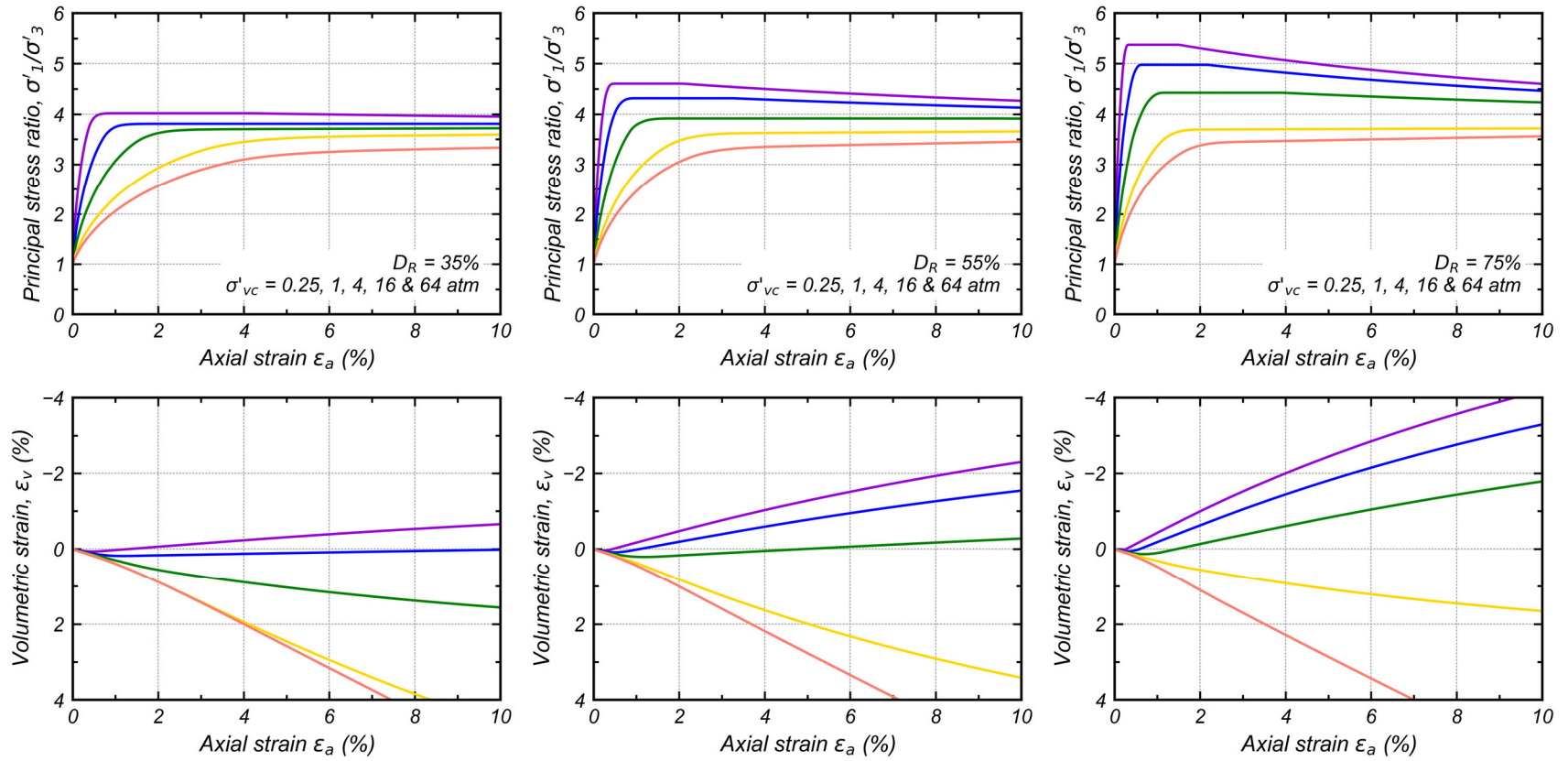


Figure 4.26. Drained monotonic PSC (plane strain compression) loading responses for $D_R = 35, 55, \text{ and } 75\%$ with initial isotropic confining stresses of $\frac{1}{4}, 1, 4, 16, \text{ and } 64 \text{ atm}$.

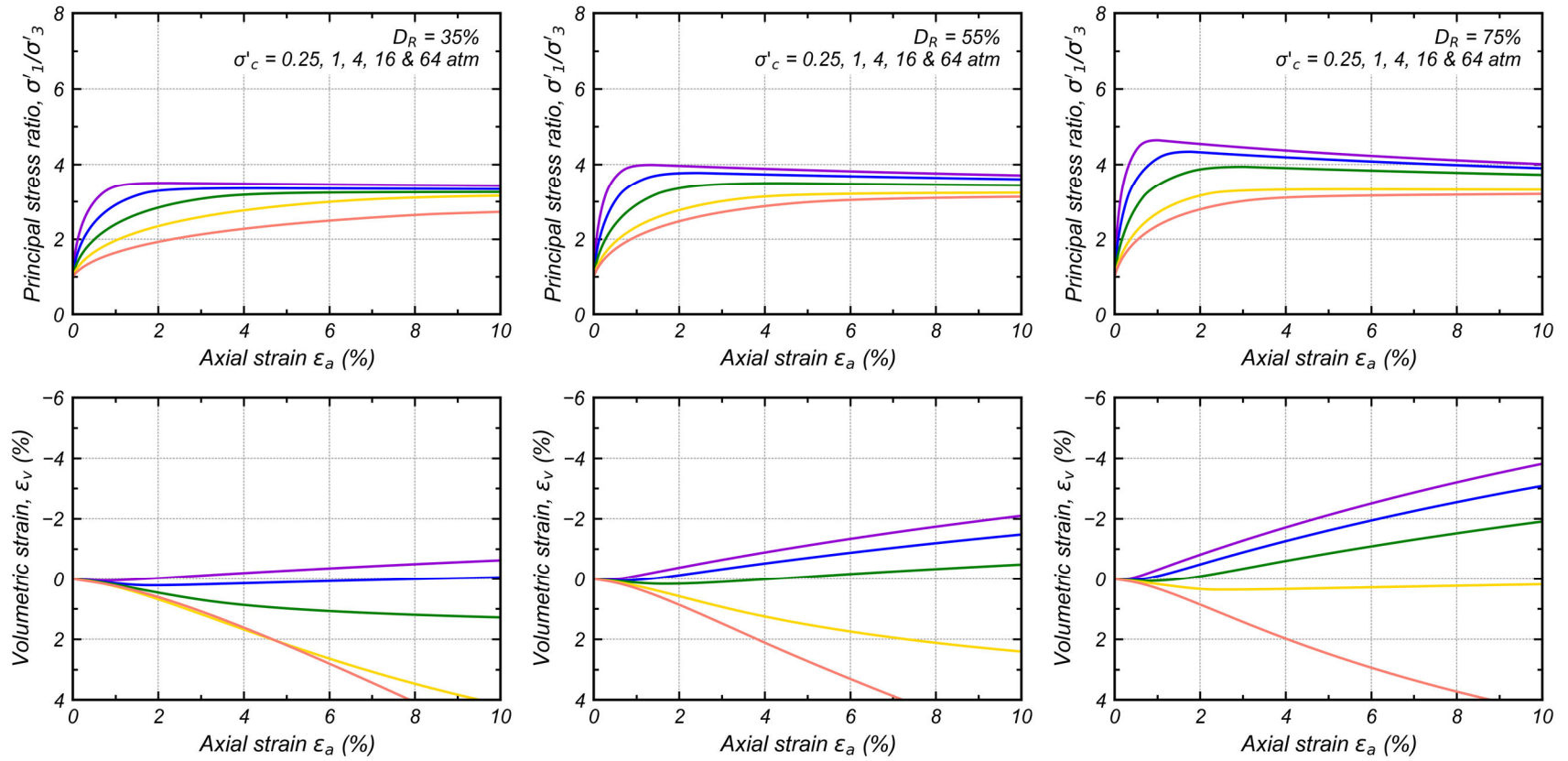


Figure 4.27. Drained monotonic TXC (triaxial compression) loading responses for $D_R = 35, 55$, and 75% with initial isotropic confining stresses of $\frac{1}{4}, 1, 4, 16$, and 64 atm.

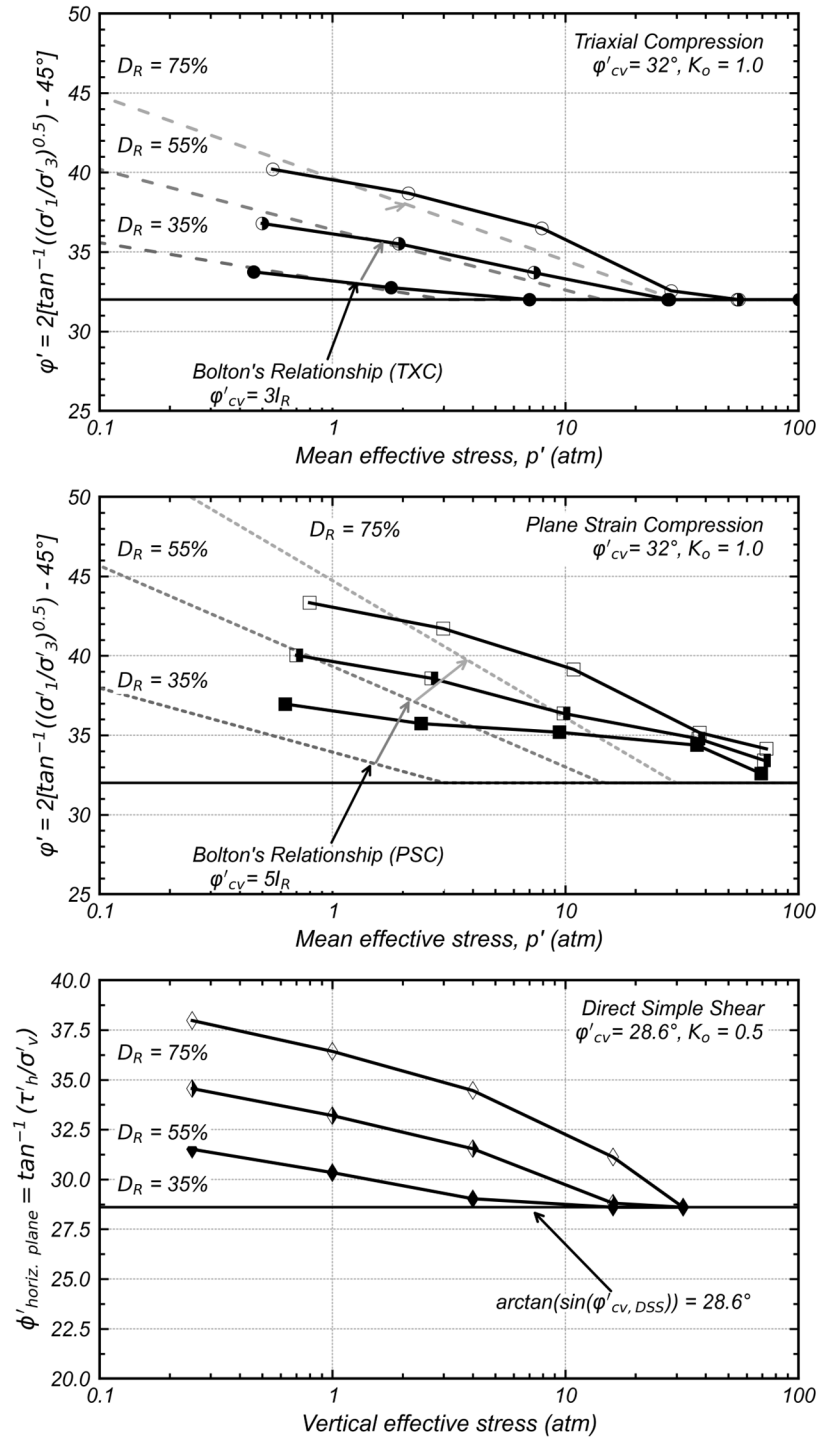


Figure 4.28. Peak friction angles from drained monotonic TxC, PSC, and DSS loading responses for $D_R = 35, 55$, and 75% under effective confining stresses of $\frac{1}{4}$, 1 , 4 , 16 , and 64 atm. For DSS loading, the friction angle is presented using the conventional interpretations that the horizontal plane is the failure plane (the actual plane of peak stress ratio is not horizontal in these simulations).

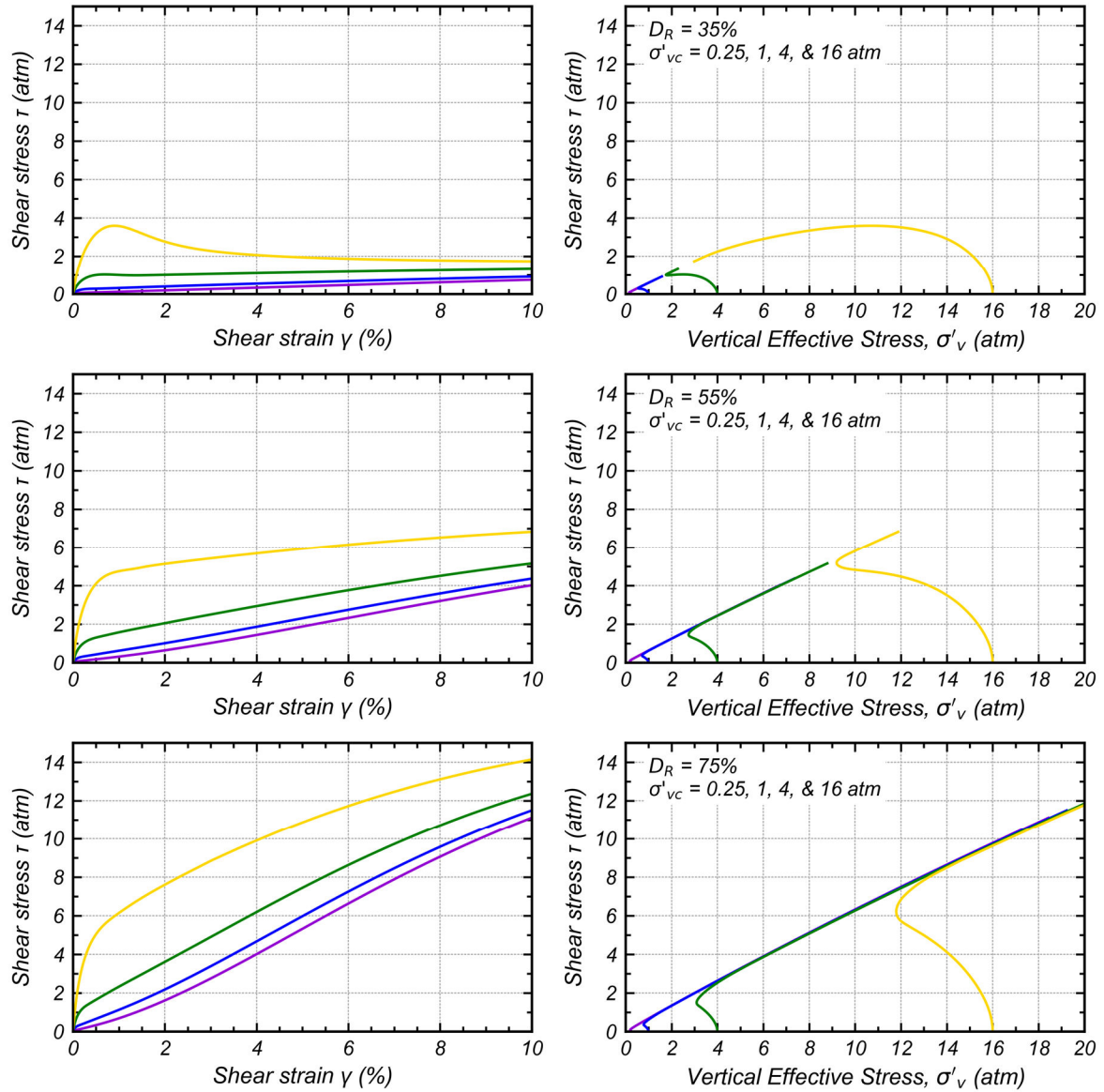


Figure 4.29. Undrained monotonic DSS loading responses for $D_R = 35, 55,$ and 75% under vertical effective consolidation stresses of $1/4, 1, 4,$ and 16 atm.

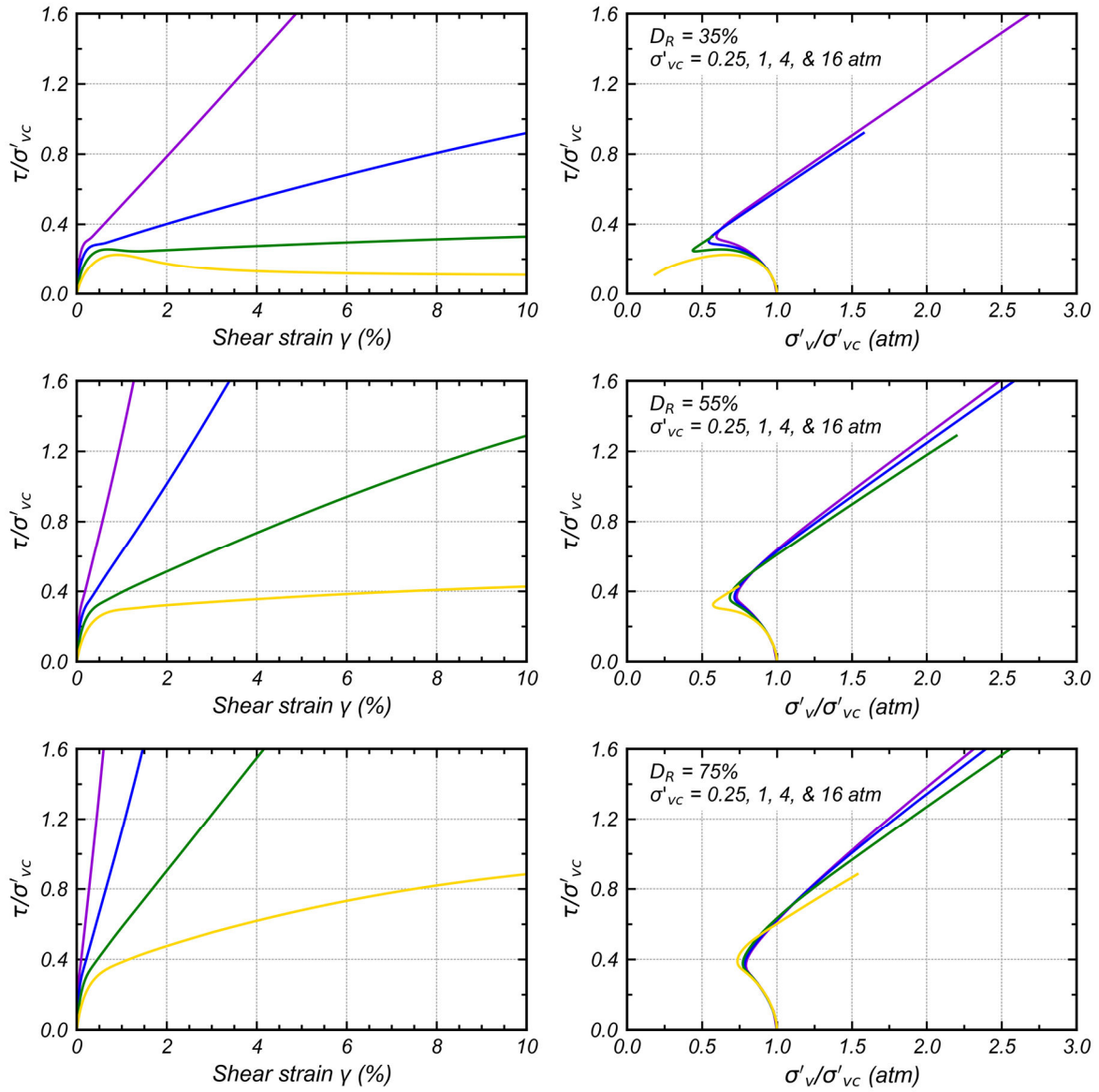


Figure 4.30. Normalized responses to undrained monotonic DSS loading for $D_R = 35$, 55 , and 75% under vertical effective consolidation stresses of $1/4$, 1 , 4 , and 16 atm.

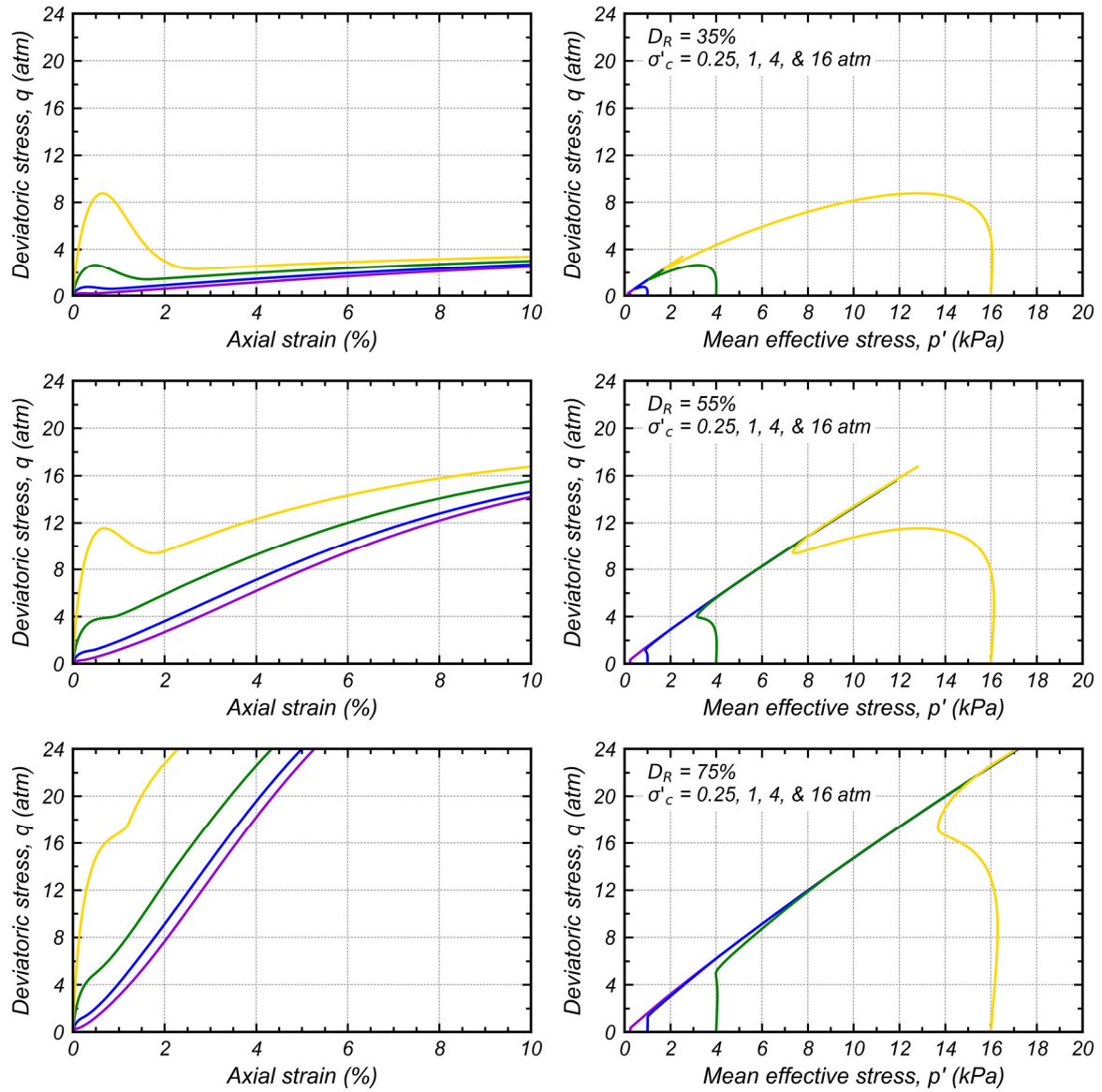


Figure 4.31. Undrained monotonic TXC loading responses for $D_R = 35, 55, \text{ and } 75\%$ under vertical effective consolidation stresses of $\frac{1}{4}, 1, 4, \text{ and } 16 \text{ atm}$.

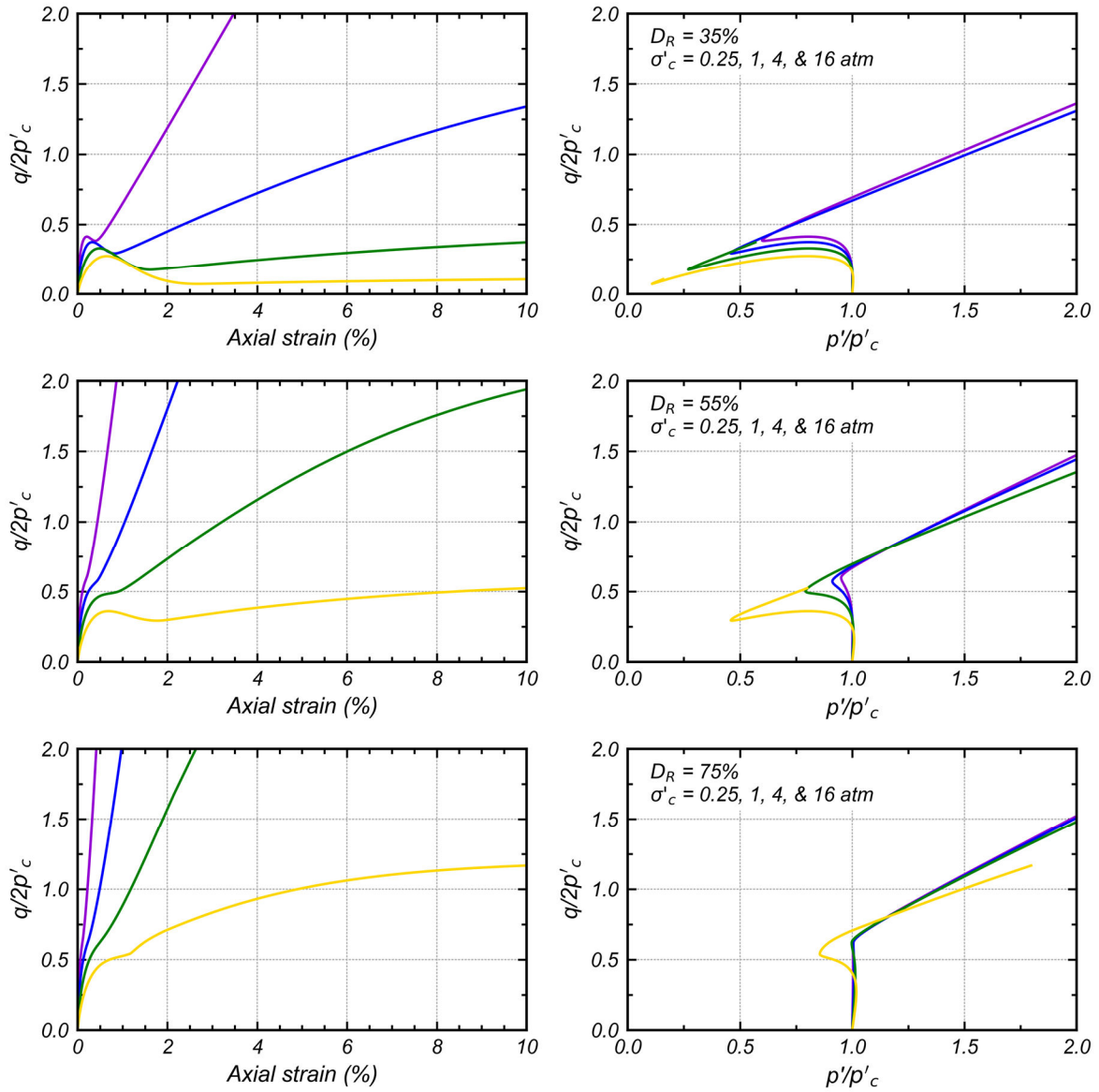


Figure 4.32. Normalized responses to undrained monotonic TXC loading for $D_R = 35, 55,$ and 75% under vertical effective consolidation stresses of $1/4, 1, 4,$ and 16 atm.

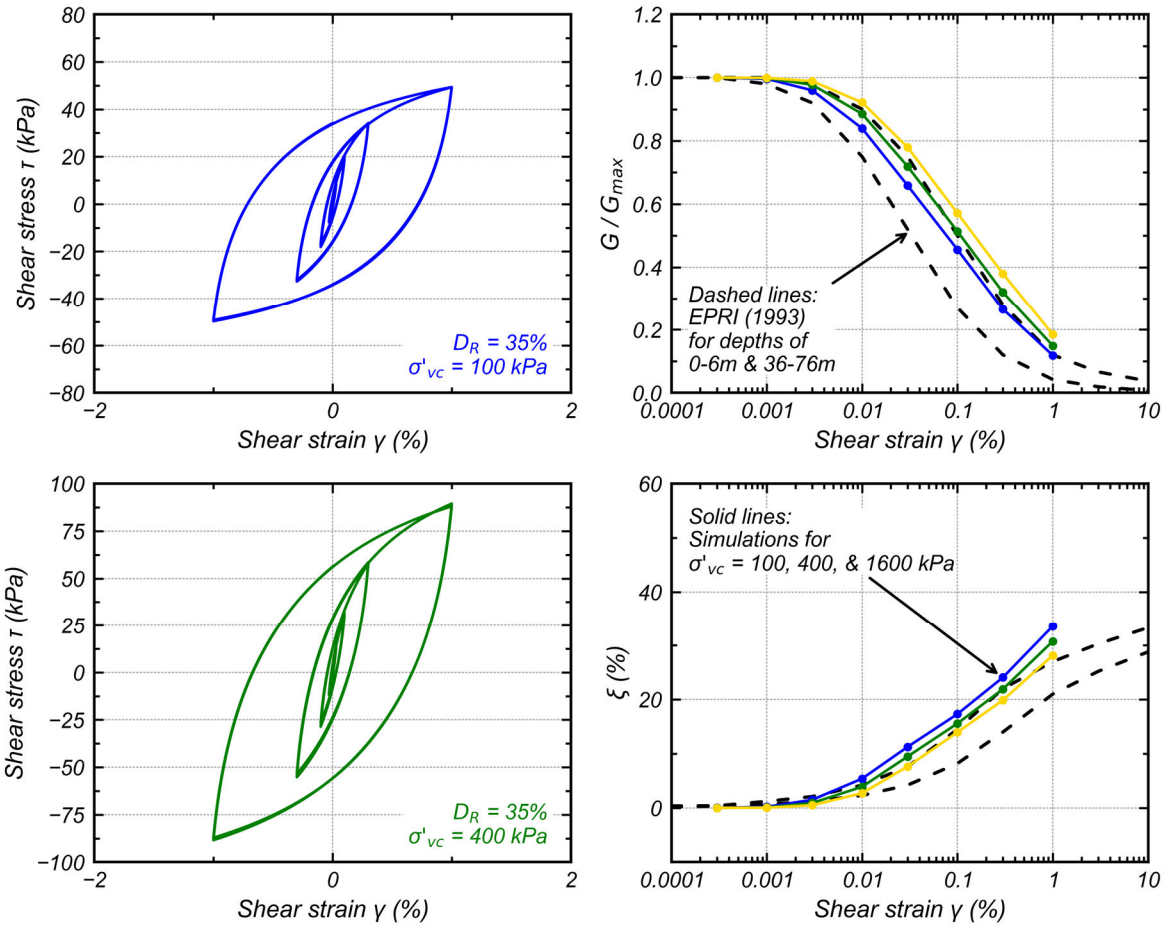


Figure 4.33. Drained strain-controlled cyclic DSS loading responses for $D_R = 35\%$ under vertical effective consolidation stresses of 1, 4, and 16 atm.

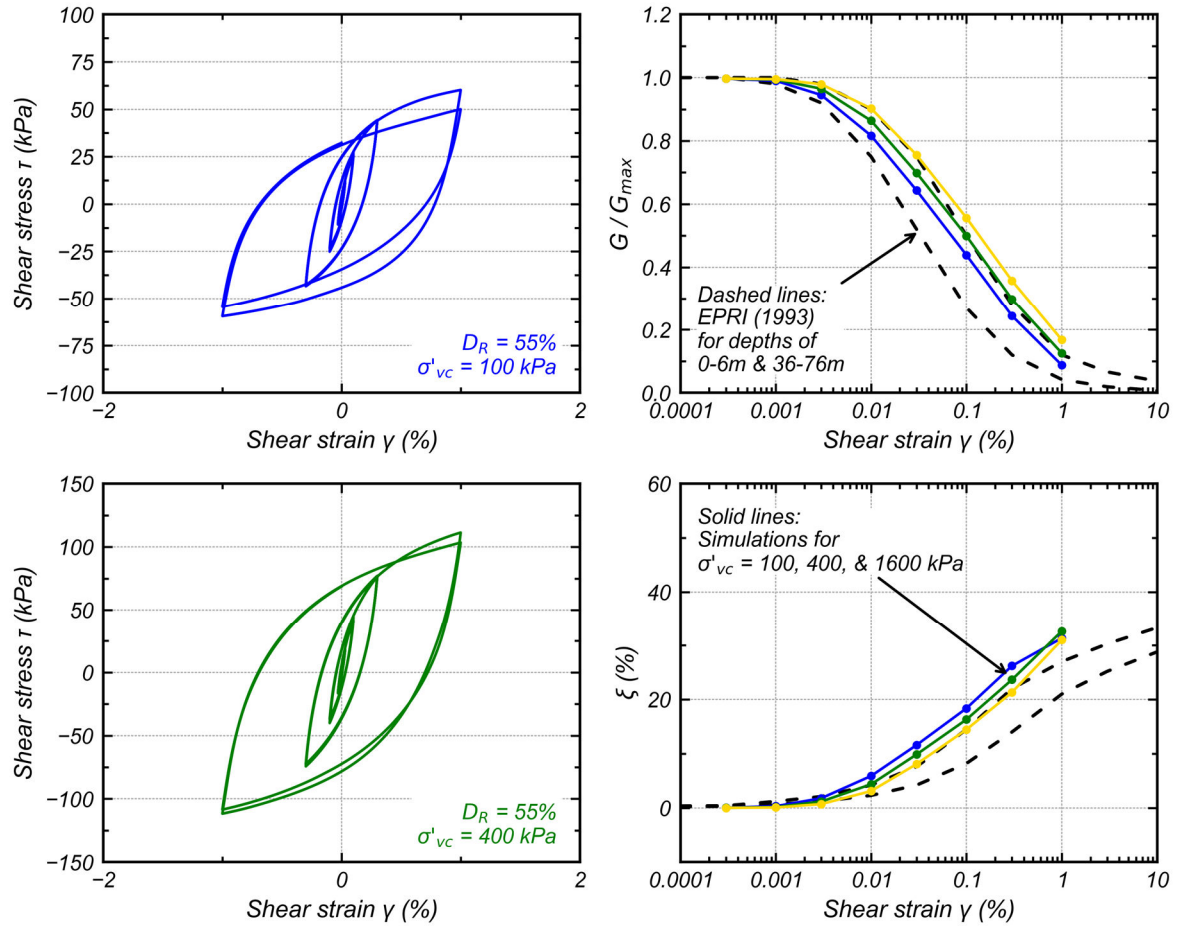


Figure 4.34. Drained strain-controlled cyclic DSS loading responses for $D_R = 55\%$ under vertical effective consolidation stresses of 1, 4, and 16 atm.

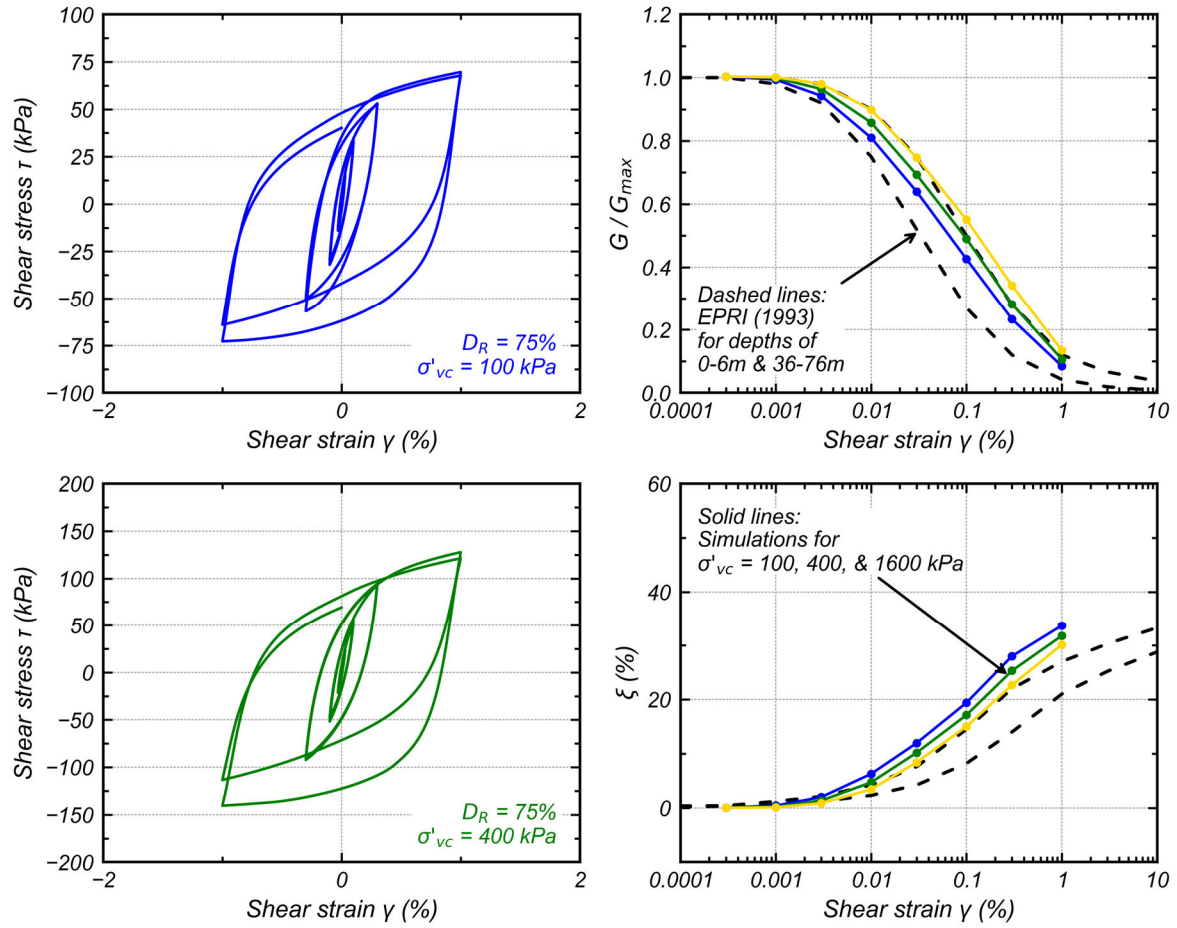


Figure 4.35. Drained strain-controlled cyclic DSS loading responses for $D_R = 75\%$ under vertical effective consolidation stresses of 1, 4, and 16 atm.

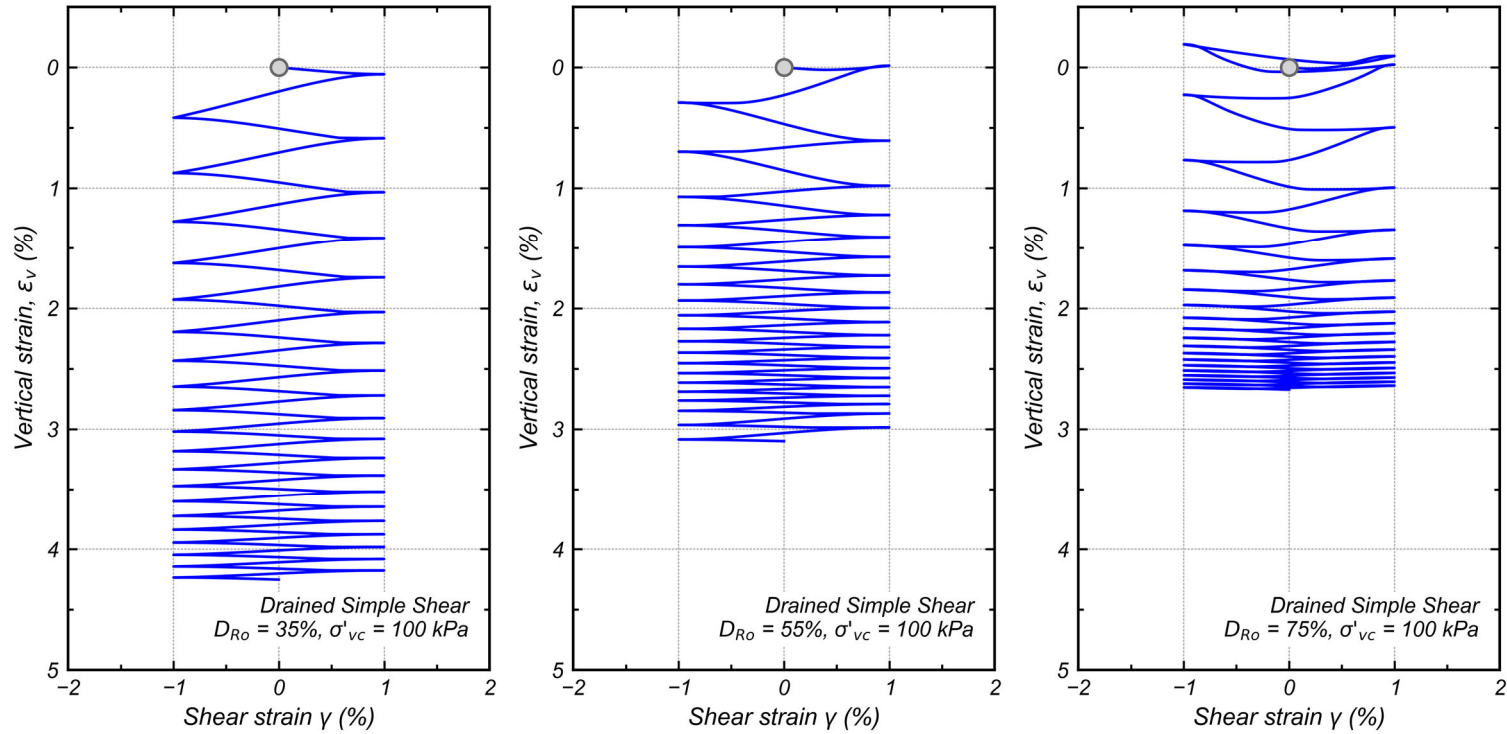


Figure 4.36. Volumetric strains during drained strain-controlled cyclic DSS loading for $D_R = 35, 55$, and 75% with a vertical effective consolidation stress of 1 atm.

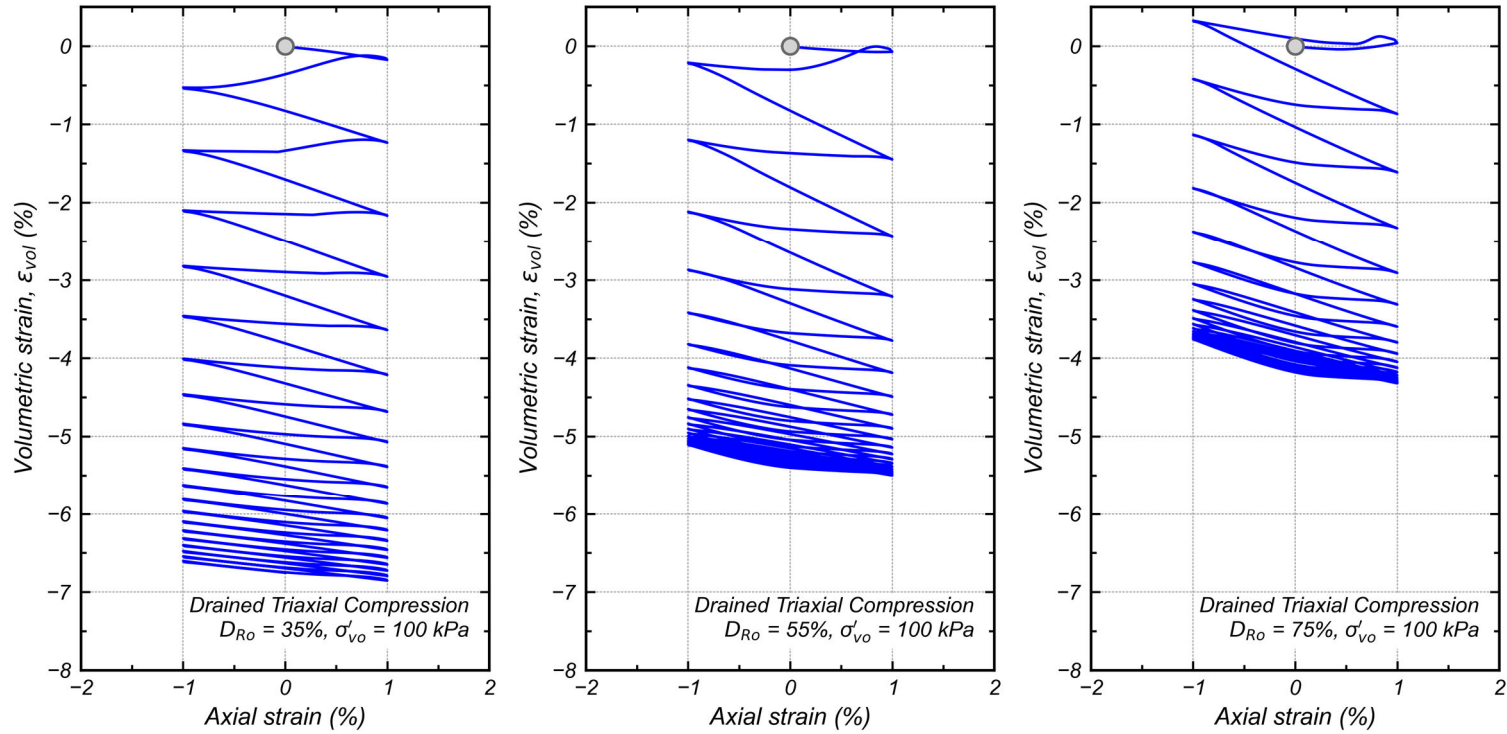


Figure 4.37. Volumetric strains during drained strain-controlled cyclic TXC loading for $D_R = 35, 55$, and 75% with a vertical effective reconsolidation stress of 1 atm.

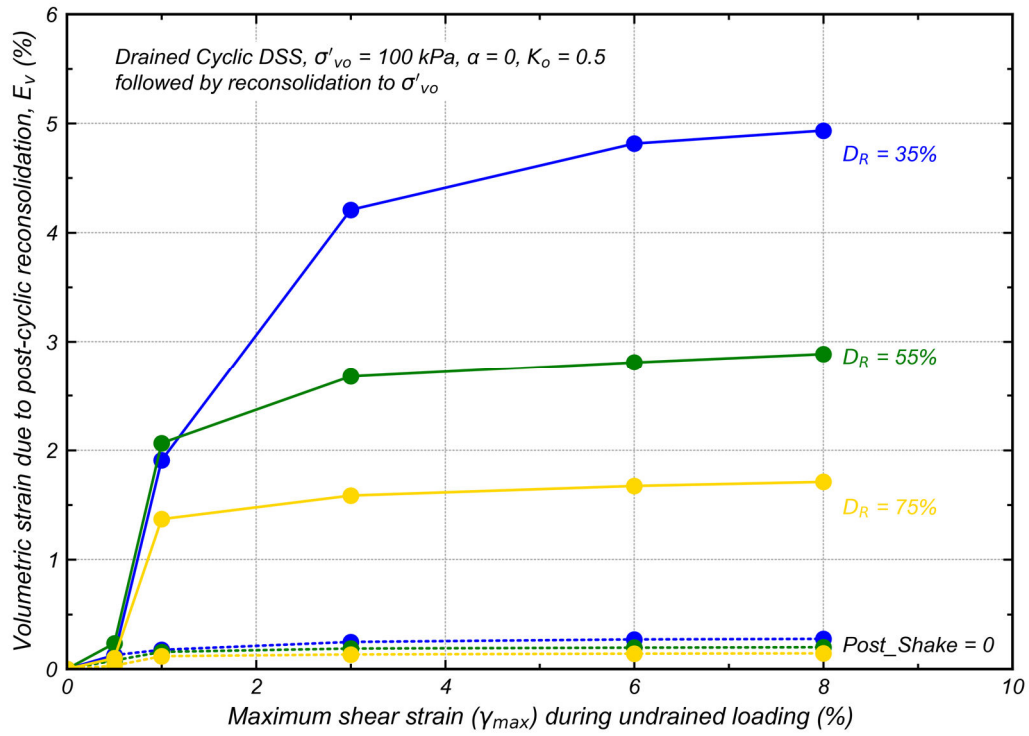


Figure 4.38. Volumetric strain due to post-cyclic reconsolidation versus the maximum shear strain induced during undrained cyclic DSS loading.

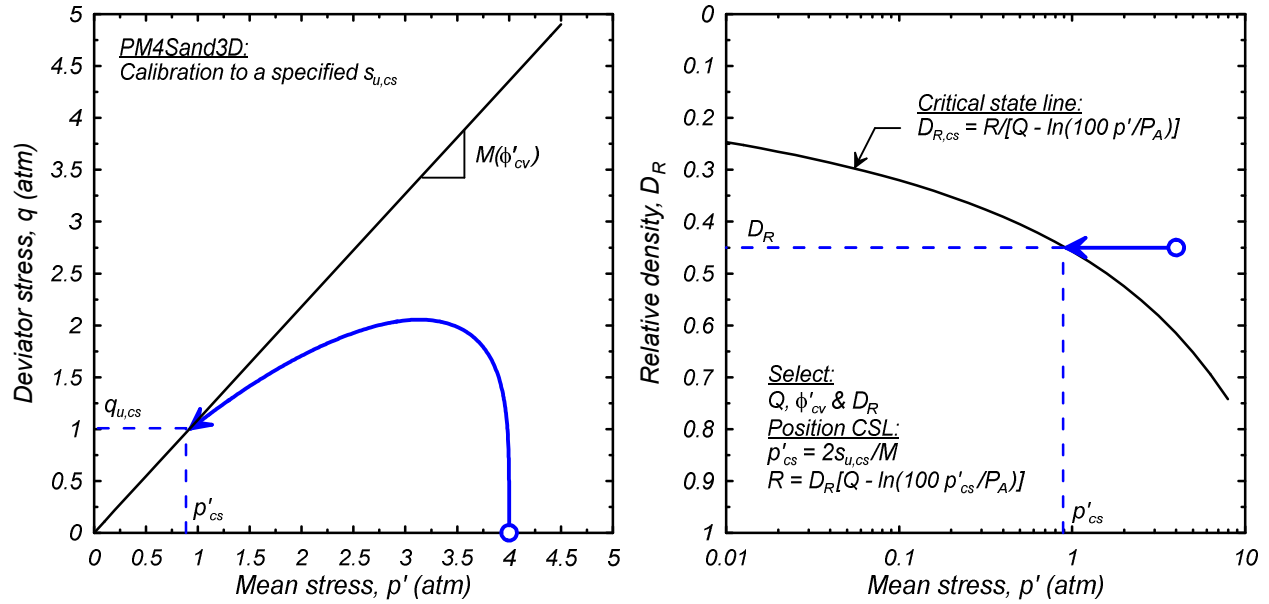


Figure 4.39. Calibrating PM4Sand to a specified undrained critical state shear strength by adjusting R (after Boulanger and Ziotopoulou 2018)

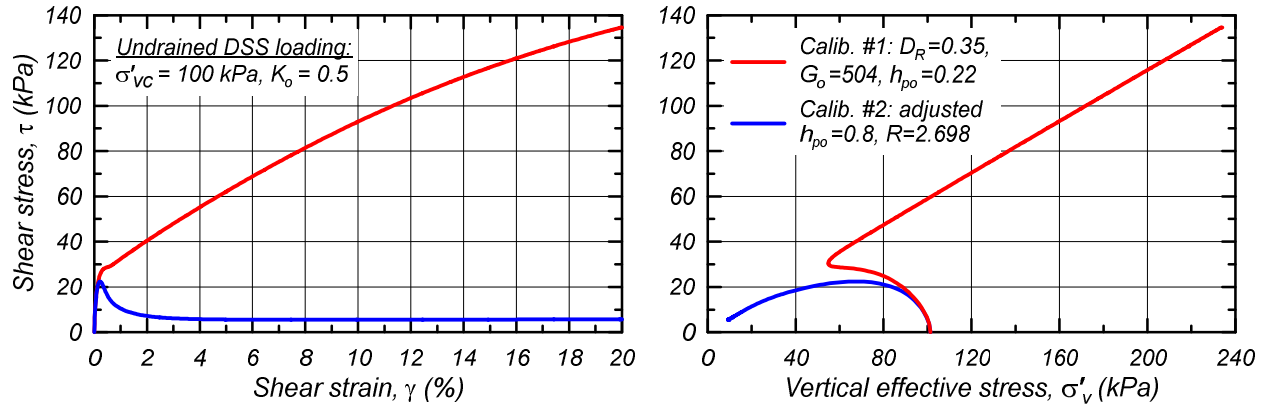


Figure 4.40. Monotonic undrained DSS responses for $D_R = 35\%$ with two calibrations: Calibration 1 with defaults for all secondary parameters, and Calibration 2 with R based on $s_{u,cs} = 7$ kPa.

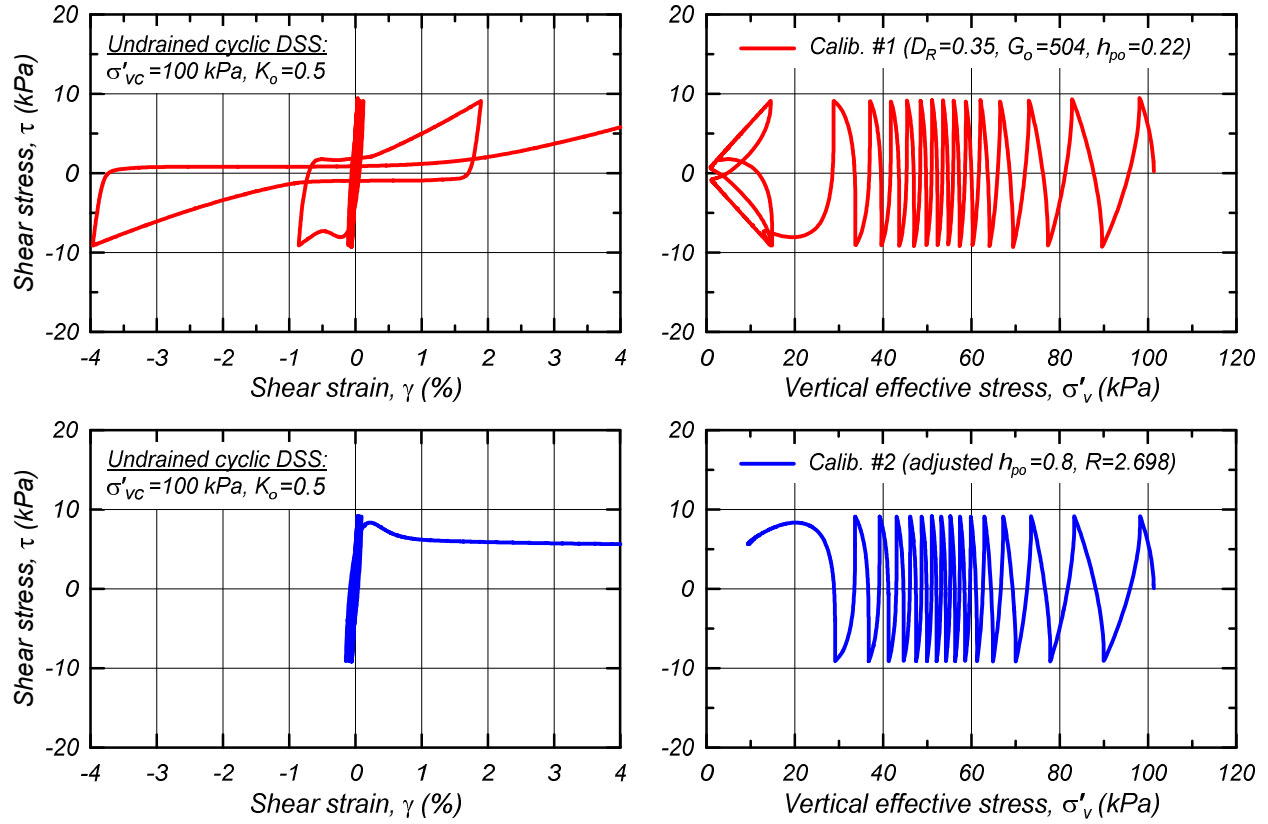


Figure 4.41. Cyclic undrained DSS response for $D_R = 35\%$ at $CSR = 0.09$ for (1) Calibration 1 with default values for all secondary parameters, and (2) Calibration 2 with R based on $s_{u,cs} = 7$ kPa.

5. CONCLUDING REMARKS

The PM4Sand3D (version 1) plasticity model presented herein is built upon the basic framework of the stress ratio-controlled, critical state-compatible, bounding surface plasticity model for sand presented by Dafalias and Manzari (2004), while incorporating the modifications and additions to the model framework made by Boulanger (2010; PM4Sand version 1), Boulanger and Ziotopoulou (2012, version 2), Boulanger and Ziotopoulou (2015, version 3), and Boulanger and Ziotopoulou (2023, version 3.3). Additional modifications were made during the transition to 3D compatibility as described herein (PM4Sand3D version 1). The changes from the Dafalias and Manzari (2004) framework were made to improve the constitutive model's ability to approximate stress-strain responses important to geotechnical earthquake engineering practice; in essence, the model was calibrated at the equation level to provide for better approximation of the trends observed in empirical correlations commonly used in practice. These modifications included:

- revising the fabric formation/destruction to depend on plastic shear rather than plastic volumetric strains;
- adding fabric history and cumulative fabric formation terms;
- modifying the plastic modulus relationship and making it dependent on fabric;
- modifying the dilatancy relationships to include dependence on fabric and fabric history, and provide more distinct control of volumetric contraction versus expansion behavior;
- providing a constraint on the dilatancy during volumetric expansion so that it is consistent with Bolton's (1986) dilatancy relationship;
- modifying the elastic modulus relationship to include dependence on stress ratio and fabric history;
- modifying the logic for tracking the initial back-stress ratio history;
- recasting the critical state framework to be in terms of a relative state parameter index;
- incorporating a methodology for improved modeling of post-liquefaction reconsolidation strains; and
- providing default values for all but three primary input parameters.

The three primary model input properties are: an apparent D_R which affects the peak drained and undrained strengths and the rate of strain accumulation during cyclic loading; the shear modulus coefficient, G_o , which should be calibrated to the estimated or measured in-situ shear wave velocity; and the contraction rate parameter, h_{po} which is used to calibrate to the estimated in-situ cyclic resistance ratio after all other properties have been set.

The model (version 1) was implemented as a user defined material in a dynamic link library for use with the commercial program FLAC3D 9.0/9.X (Itasca 2024).

The behavior of the model was illustrated by simulations of element loading tests covering a broad range of conditions, including direct simple shear and triaxial, drained and undrained, cyclic and monotonic loading under a range of initial relative densities, confining stresses, and initial shear stress conditions. The simulations presented in this report were completed using the dynamic link library (DLL) version *cmodelPM4Sand3D009.dll* compiled May 2026. The model is shown to provide reasonable approximations of desired behaviors and to be relatively easy to calibrate.

ACKNOWLEDGMENTS

The authors are indebted to Professors Bruce Kutter and Yannis Dafalias for their discussions and insights regarding the formulation and implementation of constitutive models. This model has benefited from all people who provided input on the plane-strain PM4Sand model upon which PM4Sand3D was built. The authors appreciate Itasca and BGC Engineering for providing access to FLAC3D for developing and beta testing PM4Sand3D, CMGI (Ramón Verdugo and colleagues) for feedback during beta testing of PM4Sand3D, and Arup's Richard Sturt, Amber Wang, and Kirk Ellison for proofreading the manual and providing feedback.

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