Technical Memorandum

Date: April 17, 2019

To: PFC 6.0 Documentation Set

From: David Potyondy (Itasca)

Re: Beam Contact Model [version 1]

Ref: 2-3558-01:17TM07

This memo describes the beam contact model (version 1) as provided in *PFC* 6.0.[1] A pebbled-beam (P-beam) is a string of spherical balls for which the beam contact model exists at all ball-ball contacts. A P-beam provides the structural behavior of a prismatic and bisymmetrical beam composed of isotropic, linear elastic material. The model formulation and test problem are provided in the first and second major sections, respectively. The test problem consists of a tip-loaded cantilever beam subjected to axial, flexural and twisting deformations.

[1] The beam contact model is referred to in commands and FISH by the name **`beam`** and is provided as an undocumented built-in contact model. The version number of the beam contact model is given by the command {**`contact list model-list`**} and listed in the "Minor" column.

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1.0 FORMULATION

The formulation of the beam contact model is the subject of this section. The first three subsections summarize the notational conventions, the PFC model and the kinematic variables — refer to Itasca (2019) for a complete description of these concepts. The remaining subsections contain the formulation, which begins with a definition of a P-beam, and is followed by the activity-deletion criteria, force-displacement law, properties, energies, methods and time step estimation scheme of the beam contact model.

1.1 Notational Conventions

Vectors are denoted by boldface type, such as $\mathbf{v}$. The length or magnitude of $\mathbf{v}$ is denoted $\|\mathbf{v}\|$ or simply v. The addition of a hat denotes a unit vector, such that $\hat{\mathbf{v}} = \mathbf{v}/\|\mathbf{v}\|$. The addition of a dot denotes a time derivative, such as $\dot{\mathbf{v}} = \partial\mathbf{v}/\partial t$. There is a global coordinate system (xyz). The vector $\mathbf{v}$ can be expressed in the global coordinate system by the relations:

$$\begin{aligned} &\mathbf{v} = \mathbf{v}(x, y, z) = v_x \hat{\mathbf{i}} + v_y \hat{\mathbf{j}} + v_z \hat{\mathbf{k}} \\ &\text{with } \; v_x = \mathbf{v}\cdot\hat{\mathbf{i}}, \quad v_y = \mathbf{v}\cdot\hat{\mathbf{j}}, \quad v_z = \mathbf{v}\cdot\hat{\mathbf{k}} \end{aligned} \tag{1}$$

where $\hat{\mathbf{i}}$, $\hat{\mathbf{j}}$ and $\hat{\mathbf{k}}$ are unit vectors directed along the positive x, y and z axes, respectively.

1.2 The PFC Model

The PFC programs (*PFC2D* and *PFC3D*) provide a general purpose, distinct-element modeling framework that includes a computational engine and a graphical user interface. A particular instance of the distinct-element model is referred to as a *PFC model*, which refers to both the 2D and 3D models. The PFC model simulates the movement and interaction of many finite-sized particles. The particles are rigid bodies with finite mass that move independently of one another and can both translate and rotate. Particles interact at pair-wise contacts by means of an internal force and moment. Contact mechanics are embodied in particle-interaction laws that update the internal forces and moments. The time evolution of this system is computed via the distinct-element method, which provides an explicit dynamic solution to Newton's laws of motion. The PFC model provides a synthetic material consisting of an assembly of rigid grains that interact at contacts and includes both granular and bonded materials.

We here generalize and expand upon the definition of the PFC model given above. The PFC model simulates the movement of particles and their mechanical interaction at pair-wise contacts. We denote each particle as a *body* to clarify that it is not a point mass but, instead, is a rigid body with finite mass and a well-defined surface. The PFC model consists of bodies and *contacts* (see Figure 1). There are three types of bodies: *balls*, *clumps* and *walls*. Bodies have surface properties that are assigned to the *pieces* on the body surface. A ball consists of one piece, which is the ball itself, while the pieces of a clump and wall are called *pebbles* and *facets*, respectively. A ball is a rigid unit-thickness disk in 2D or sphere in 3D. A clump is a collection of pebbles that are rigid unit-thickness disks in 2D or spheres in 3D. Clumps model arbitrarily shaped rigid bodies. The pebbles comprising a clump can overlap but contacts do not exist between them; instead, contacts form between the pebbles on the boundary of a clump and other bodies. A wall is a collection of facets that are linear segments in 2D or triangles in 3D and that form a manifold and orientable surface.

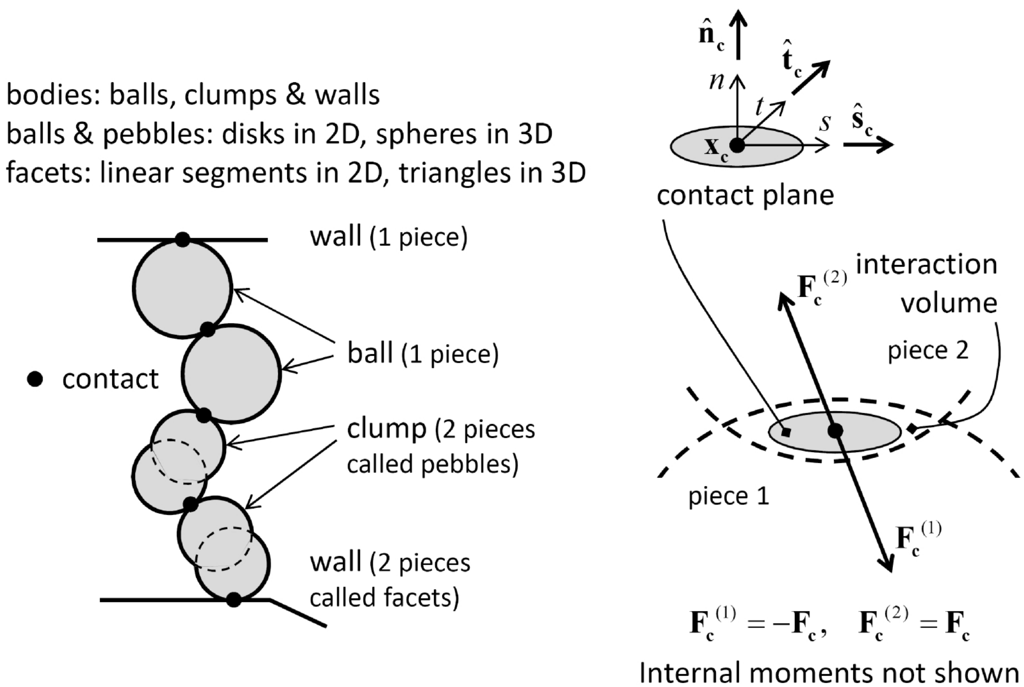

Figure 1 ***PFC model showing bodies and contacts (left) and contact plane with internal force (right).*** (From Fig. 1 of Itasca [2019][2].)

Contact mechanics are embodied in particle-interaction laws that employ a soft-contact approach for which all deformation occurs at the contacts between the rigid bodies. The mechanical interaction between the surfaces of two bodies occurs at one or more pair-wise mechanical contacts. Contacts are created and deleted based on body proximity by the contact-detection logic. A contact provides an interface between two pieces. The interface consists of a contact plane with location $(\mathbf{x}_c)$, normal direction $(\hat{\mathbf{n}}_c)$, and coordinate system (nst). The contact plane is centered within the interaction volume (either gap or overlap) of the two pieces, oriented tangential to the two pieces, and rotated to ensure that relative motion of the piece surfaces remains symmetric with respect to the contact plane. Each contact stores a force $(\mathbf{F}_c)$ and moment $(\mathbf{M}_c)$ that act at the contact location in

[2] In documentation set at PFC: Numerical Simulations with PFC: PFC Model Formulation: Model Components.

an equal and opposite sense on the two pieces. The internal force and moment are updated by the particle-interaction law, which takes the relative motion and surface properties of the two pieces as input. We refer to the particle-interaction law as a *contact model.*

1.3 Kinematic Variables

Kinematics considers the motion of systems of bodies without regard to the role of the forces causing the motion, while kinetics considers the relationship of the forces to the kinematic variables. The kinetics of the PFC model are embodied in the force-displacement law of each contact model. The kinematic variables that serve as the input to the force-displacement law are discussed here.

Contact resolution occurs when a new contact is detected during the cycle sequence, prior to the force-displacement calculations. During contact resolution, the contact state variables (see Table 1) are updated. Each contact model uses its properties, along with the relative motion of the two contacting pieces, to update the contact force and moment.

Table 1 Contact State Variables

Property	Description
m_c	effective inertial mass
Contact plane (see Figures 1 and 2):	
$\mathbf{x_c}$	contact-plane location
$\hat{\mathbf{n}}_\mathbf{c}$	contact-plane normal direction
$\hat{\mathbf{s}}_\mathbf{c}$	contact-plane coordinate system (s axis)
$\hat{\mathbf{t}}_\mathbf{c}$	contact-plane coordinate system (t axis)
g_c	contact gap ($g_c > 0$ is open)
Relative motion (see Figures 3 and 4):	
$\dot{\boldsymbol{\delta}}$	relative translational velocity
$\dot{\boldsymbol{\theta}}$	relative rotational velocity
$\Delta\delta_n$	relative normal-displacement increment ($\Delta\delta_n > 0$ is opening)
$\Delta\boldsymbol{\delta}_\mathbf{s}$	relative shear-displacement increment
$\Delta\theta_t$	relative twist-rotation increment

$\Delta\boldsymbol{\theta}_{\mathbf{b}}$	relative bend-rotation increment $(\Delta\theta_{bs}, \Delta\theta_{bt})$

The contact shown in Figure 2 has been created between the pieces of two bodies. Each contact has two ends, **end1** and **end2**, with the associated pieces and bodies labelled 1 and 2. The bodies are rigid. Therefore, the motion of body (b) is described by its rotational velocity $\left(\boldsymbol{\omega}^{(b)}\right)$ and the translational velocity $\left(\dot{\mathbf{x}}^{(b)}\right)$ of its centroid $\left(\mathbf{x}^{(b)}\right)$. The contact state variables include the contact-plane information as well as the contact gap (g_c), which is the minimal signed distance separating the piece surfaces. A vector quantity that lies on the contact plane ($\mathbf{S}$) can be expressed in the contact plane coordinate system by the relations:

$$
\begin{aligned}
&\mathbf{S} = \mathbf{S}(s,t) = S_s\hat{\mathbf{s}}_{\mathbf{c}} + S_t\hat{\mathbf{t}}_{\mathbf{c}} \\
&\text{with } \; S_s = \mathbf{S}\cdot\hat{\mathbf{s}}_{\mathbf{c}}, \quad S_t = \mathbf{S}\cdot\hat{\mathbf{t}}_{\mathbf{c}}.
\end{aligned}
\tag{2}
$$

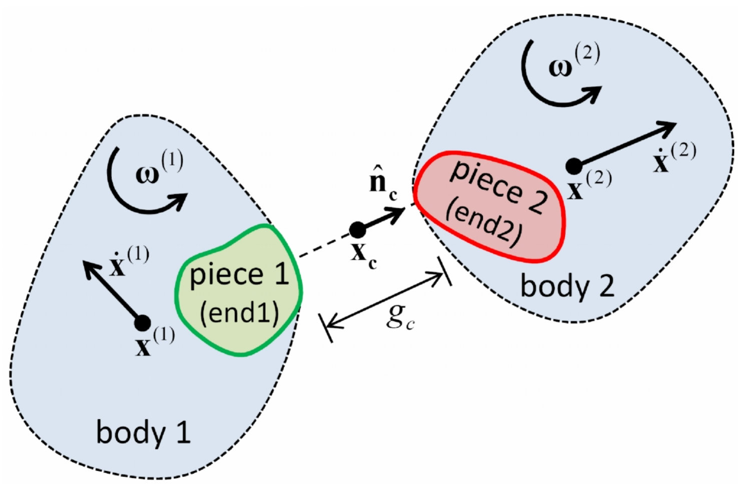

Figure 2 ***A contact between the pieces of two bodies.*** (From Fig. 1 of Itasca [2019][3].)

The relative motion of the piece surfaces at a contact is described by the relative translational $(\dot{\boldsymbol{\delta}})$ and rotational $(\dot{\boldsymbol{\theta}})$ velocities:

$$
\begin{aligned}
\dot{\boldsymbol{\delta}} &= \dot{\mathbf{x}}_{\mathbf{c}}^{(2)} - \dot{\mathbf{x}}_{\mathbf{c}}^{(1)} \\
\dot{\boldsymbol{\theta}} &= \boldsymbol{\omega}^{(2)} - \boldsymbol{\omega}^{(1)}.
\end{aligned}
\tag{3}
$$

[3] In documentation set at PFC: PFC Model Objects: Contacts and Contact Models: Contact Resolution.

In this expression, $\dot{\mathbf{x}}_{\mathbf{c}}^{(b)}$ is the translational velocity of body (b) at the contact location:

$$\dot{\mathbf{x}}_{\mathbf{c}}^{(b)} = \dot{\mathbf{x}}^{(b)} + \boldsymbol{\omega}^{(b)} \times \left(\mathbf{x}_{\mathbf{c}} - \mathbf{x}^{(b)} \right) \tag{4}$$

where $\dot{\mathbf{x}}^{(b)}$ is the translational velocity of body (b); $\boldsymbol{\omega}^{(b)}$ is the rotational velocity of body (b); $\mathbf{x}_{\mathbf{c}}$ is the contact location; and $\mathbf{x}^{(b)}$ is either the centroid (if the body is a ball or clump) or the center of rotation (if the body is a wall) of body (b). The contact location defines a point that is fixed with respect to each body, and thus, $\dot{\mathbf{x}}_{\mathbf{c}}^{(b)}$ is the translational velocity of that point in body (b).[4]

The relative translational velocity can be expressed as

$$\begin{aligned} &\dot{\boldsymbol{\delta}} = \dot{\boldsymbol{\delta}}_{\mathbf{n}} + \dot{\boldsymbol{\delta}}_{\mathbf{s}} \\ &\text{with} \quad \dot{\boldsymbol{\delta}}_{\mathbf{n}} = \left(\dot{\boldsymbol{\delta}} \cdot \hat{\mathbf{n}}_{\mathbf{c}} \right) \hat{\mathbf{n}}_{\mathbf{c}} = \dot{\delta}_n \hat{\mathbf{n}}_{\mathbf{c}}, \quad \dot{\boldsymbol{\delta}}_{\mathbf{s}} = \dot{\boldsymbol{\delta}} - \dot{\boldsymbol{\delta}}_{\mathbf{n}} \end{aligned} \tag{5}$$

where $\dot{\boldsymbol{\delta}}_{\mathbf{n}}$ ($\dot{\delta}_n > 0$ is moving apart) and $\dot{\boldsymbol{\delta}}_{\mathbf{s}}$ are the relative translational velocities normal and tangential, respectively, to the contact plane, and the subscripts n and s correspond with normal and shear action, respectively (see Figure 3 — the centering of the contact within the interaction volume ensures that the relative displacement is symmetric with respect to the contact plane).

The relative rotational velocity can be expressed as

$$\begin{aligned} &\dot{\boldsymbol{\theta}} = \dot{\boldsymbol{\theta}}_{\mathbf{t}} + \dot{\boldsymbol{\theta}}_{\mathbf{b}} \\ &\text{with} \quad \dot{\boldsymbol{\theta}}_{\mathbf{t}} = \left(\dot{\boldsymbol{\theta}} \cdot \hat{\mathbf{n}}_{\mathbf{c}} \right) \hat{\mathbf{n}}_{\mathbf{c}} = \dot{\theta}_t \hat{\mathbf{n}}_{\mathbf{c}}, \quad \dot{\boldsymbol{\theta}}_{\mathbf{b}} = \dot{\boldsymbol{\theta}} - \dot{\boldsymbol{\theta}}_{\mathbf{t}} \end{aligned} \tag{6}$$

[4] The location of this point within each body may change — e.g., under increasing applied compression, the overlap increases and these points move deeper into each body.

where $\dot{\boldsymbol{\theta}}_{\mathbf{t}}$ and $\dot{\boldsymbol{\theta}}_{\mathbf{b}}$ are the relative rotational velocities normal and tangential, respectively, to the contact plane, and the subscripts t and b correspond with twisting and bending action, respectively (see Figure 4).

The relative displacement and rotation increments at the contact during a time step Δt are

$$\begin{aligned}
&\Delta\boldsymbol{\delta} = \Delta\delta_n \hat{\mathbf{n}}_{\mathbf{c}} + \Delta\boldsymbol{\delta}_{\mathbf{s}} \;\left[\Delta\delta_{ss} = \Delta\boldsymbol{\delta}_{\mathbf{s}} \cdot \hat{\mathbf{s}}_{\mathbf{c}},\; \Delta\delta_{st} = \Delta\boldsymbol{\delta}_{\mathbf{s}} \cdot \hat{\mathbf{t}}_{\mathbf{c}}\right] \\
&\Delta\boldsymbol{\theta} = \Delta\theta_t \hat{\mathbf{n}}_{\mathbf{c}} + \Delta\boldsymbol{\theta}_{\mathbf{b}} \;\left[\Delta\theta_{bs} = \Delta\boldsymbol{\theta}_{\mathbf{b}} \cdot \hat{\mathbf{s}}_{\mathbf{c}},\; \Delta\theta_{bt} = \Delta\boldsymbol{\theta}_{\mathbf{b}} \cdot \hat{\mathbf{t}}_{\mathbf{c}}\right] \\
&\text{with} \quad \Delta\delta_n = \dot{\delta}_n \Delta t, \quad \Delta\boldsymbol{\delta}_{\mathbf{s}} = \dot{\boldsymbol{\delta}}_{\mathbf{s}} \Delta t \\
&\qquad\quad \Delta\theta_t = \dot{\theta}_t \Delta t, \quad \Delta\boldsymbol{\theta}_{\mathbf{b}} = \dot{\boldsymbol{\theta}}_{\mathbf{b}} \Delta t
\end{aligned} \tag{7}$$

where $\Delta\delta_n$ is the relative normal-displacement increment, $\Delta\boldsymbol{\delta}_{\mathbf{s}}$ is the relative shear-displacement increment, $\Delta\theta_t$ is the relative twist-rotation increment, and $\Delta\boldsymbol{\theta}_{\mathbf{b}}$ is the relative bend-rotation increment.

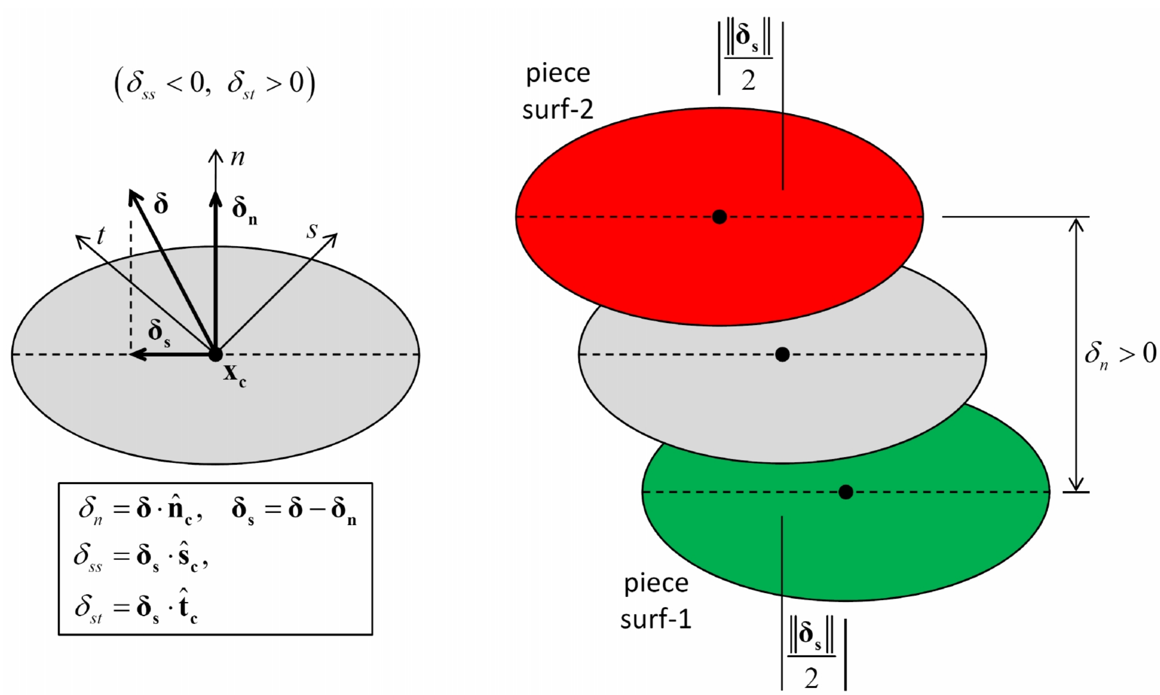

Figure 3 ***Kinematics of a contact showing contact plane with relative displacement and motion of piece surfaces.*** (From Fig. 7 of Itasca [2019][5].)

[5] In documentation set at PFC: PFC Model Objects: Contacts and Contact Models: Contact Resolution.

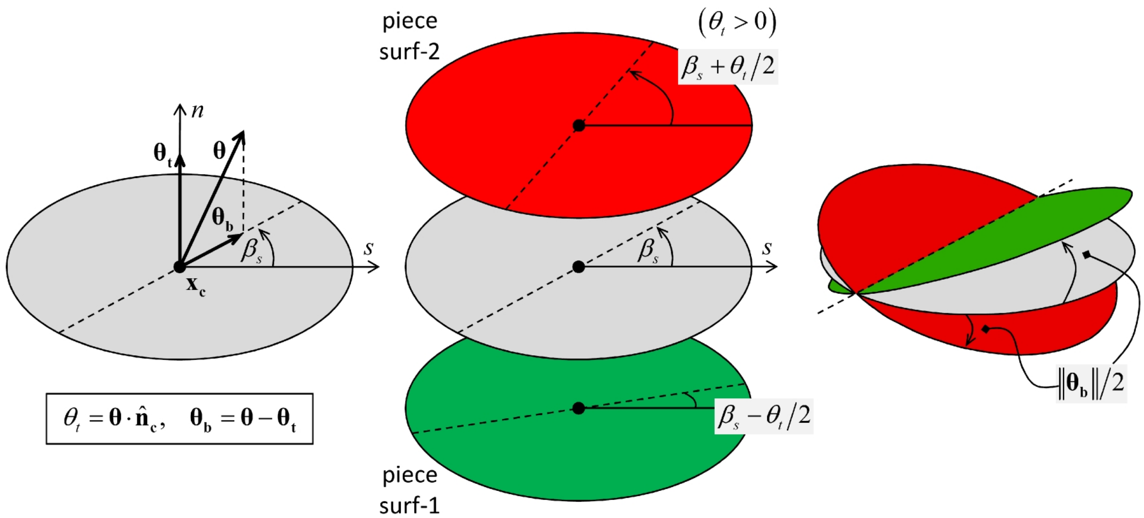

Figure 4 Kinematics of a contact showing contact plane with relative rotation and motion of piece surfaces. (From Fig. 8 of Itasca [2019][6].)

1.4 P-Beam

A pebbled beam (P-beam) is defined as a string of spherical balls for which the beam contact model exists at all ball-ball contacts. A P-beam provides the structural behavior of a prismatic and bisymmetrical beam composed of isotropic, linear elastic material. A P-beam is defined by the string of balls, and the ball densities (used along with ball radii to obtain ball mass). The structural properties of a P-beam are defined by the properties of each beam contact model in the string. The properties of the beam contact model are listed in Section 1.7.

There are two coordinate systems associated with the interface of the beam contact model (see Figure 5): the contact plane coordinate system (*nst*), and the beam coordinate system (*xyz*). The beam cross-sectional properties are specified in the beam coordinate system. The beam coordinate system is defined by the centers of the two contacting balls and by the vector $\mathbf{Y}$ such that: (1) the centroidal axis coincides with the *x*-axis; (2) the *x*-axis is directed from the center of ball-1 to the center of ball-2; and (3) the *y*-axis is aligned with the projection of $\mathbf{Y}$ onto the cross-sectional plane. If $\mathbf{Y}$ is not specified, or is parallel with the local *x*-axis, then $\mathbf{Y}$ defaults to the global *y*- or *x*-direction, whichever is not parallel with the local *x*-axis. For the general beam cross section shown in Figure 5, the cross-sectional properties of area (A), polar moment of inertia (J), and moments of inertia about the y- and z-axes (I_y and I_z) are defined by the integrals:

$$A = \int_A dA, \;\; J = \int_A r^2 dA, \;\; I_y = \int_A z^2 dA, \;\; I_z = \int_A y^2 dA \tag{8}$$

[6] In documentation set at PFC: PFC Model Objects: Contacts and Contact Models: Contact Resolution.

where the two principal axes of the beam cross section are defined by the y- and z-axes. The cross-sectional properties of the rectangular cross section shown in Figure 5 are given by (Ugural and Fenster, 1987, Appendix C)

$$A = bh, \quad J = \frac{bh\left(b^2 + h^2\right)}{12}, \quad I_y = \tfrac{1}{12} bh^3, \quad I_z = \tfrac{1}{12} hb^3. \tag{9}$$

The cross-sectional properties are specified directly (**`bm_bhGiven = 0`**), or the cross section is assumed to be rectangular (**`bm_bhGiven = 1`**) with specified width (b) and height (h), and the cross-sectional properties are computed internally via Eq. (9).

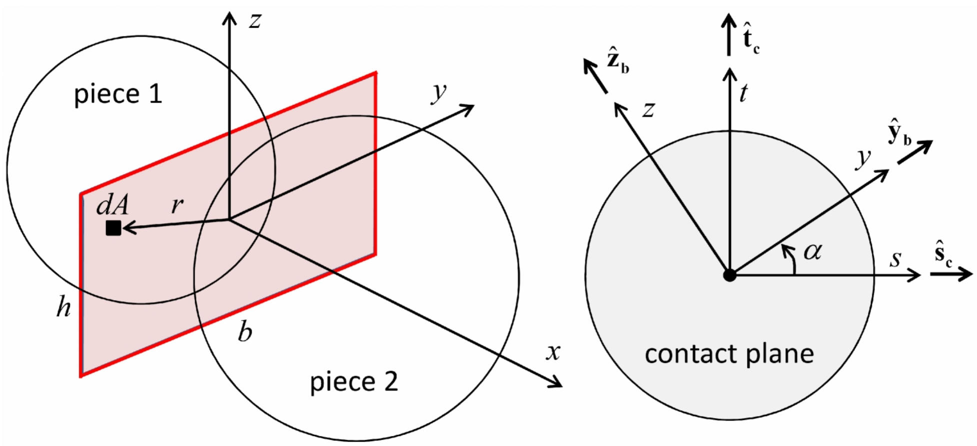

Figure 5 ***Beam coordinate system (xyz) with the two principal axes of the beam cross section defined by the y- and z-axes (left). Relationship between beam coordinate system and contact plane coordinate system (right).***

The beam coordinate system should be defined before cycling by specifying the **`bm_Ydir`** property. The beam coordinate system remains fixed with respect to the contact plane coordinate system as shown in Figure 5. A beamed contact is a contact that has been assigned the beam contact model. During the first cycle after a beamed contact has been created, $\mathbf{Y}$ is projected onto the contact plane to obtain $\hat{\mathbf{y}}_\mathrm{b}$, which is oriented at an angle α with respect to the s-direction. The contact model does not store α, instead it stores the cosine and sine of α given by

$$\cos\alpha = \hat{\mathbf{y}}_\mathrm{b} \cdot \hat{\mathbf{s}}_\mathrm{c}, \quad \sin\alpha = \hat{\mathbf{y}}_\mathrm{b} \cdot \hat{\mathbf{t}}_\mathrm{c}. \tag{10}$$

These terms are used to update the bending moments in the x- and y-directions (see the Force-Displacement Law). The mapping between the beam and contact coordinate systems of a vector $\mathbf{S}$ that lies on the interface is given by the relations:

$$\begin{Bmatrix} S_y \\ S_z \end{Bmatrix} = \left[\mathbf{t_{gl}}\right] \begin{Bmatrix} S_s \\ S_t \end{Bmatrix}, \quad \begin{Bmatrix} S_s \\ S_t \end{Bmatrix} = \left[\mathbf{t_{gl}}\right]^T \begin{Bmatrix} S_y \\ S_z \end{Bmatrix}$$
$$\left[\mathbf{t_{gl}}\right] = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \tag{11}$$
$$y_s = \hat{\mathbf{y}}_\mathbf{b} \cdot \hat{\mathbf{s}}_\mathbf{c} = \cos\alpha, \quad y_t = \hat{\mathbf{y}}_\mathbf{b} \cdot \hat{\mathbf{t}}_\mathbf{c} = \sin\alpha$$
$$z_s = \hat{\mathbf{z}}_\mathbf{b} \cdot \hat{\mathbf{s}}_\mathbf{c} = -\sin\alpha, \quad z_t = \hat{\mathbf{z}}_\mathbf{b} \cdot \hat{\mathbf{t}}_\mathbf{c} = \cos\alpha.$$

The beam interfaces are displayed as a microstructural plot set (see Section 3.2 in Potyondy [2019]). The beam interfaces plot set (geometry set name: **"beam interfaces"**) displays the interface of each beamed contact. The interface and *y*-axis can be displayed by the geometry plot item by specifying {Colorby: Group} and {Sets: beam interfaces}, with {Colors:} used to turn each entity on/off and specify its color. The interface is drawn as a rectangular polygon with width and height of the beam. The beam *y*-axis is drawn as a line from the interface center that extends just beyond the interface edge. If the width and height are not specified for the beam, then only the *y*-axis is drawn.

1.5 Activity-Deletion Criteria

A contact with the beam model is always active.

1.6 Force-Displacement Law

The force-displacement law for the beam contact model updates the contact force and moment:

$$\mathbf{F_c} = \mathbf{F}, \quad \mathbf{M_c} = \mathbf{M} \tag{12}$$

where $\mathbf{F}$ is the beam force and $\mathbf{M}$ is the beam moment. The beam force is resolved into a normal and shear force, and the beam moment is resolved into a twisting and bending moment:

$$\mathbf{F} = -F_n \hat{\mathbf{n}}_\mathbf{c} + \mathbf{F_s}, \quad \mathbf{M} = M_t \hat{\mathbf{n}}_\mathbf{c} + \mathbf{M_b} \tag{13}$$

where $F_n > 0$ is tension. The beam shear force and bending moment lie on the contact plane and are expressed in the contact plane coordinate system:

$$\mathbf{F_s} = F_{ss} \hat{\mathbf{s}}_\mathbf{c} + F_{st} \hat{\mathbf{t}}_\mathbf{c}, \quad \mathbf{M_b} = M_{bs} \hat{\mathbf{s}}_\mathbf{c} + M_{bt} \hat{\mathbf{t}}_\mathbf{c}. \tag{14}$$

When a beamed contact is created, an interface between two notional surfaces is established, and the beam force and moment are zeroed. The beamed contact provides an elastic interaction between these two notional surfaces. Each notional surface is connected rigidly to its associated ball. The beam surface gap is defined as the cumulative relative normal displacement of the ball surfaces:

$$g_s = \sum \Delta\delta_n \tag{15}$$

where $\Delta\delta_n$ is the relative normal-displacement increment of Eq. (7).

The following internal state variables are updated during the first cycle. The beam length

$$L = \left\| \mathbf{x}^{(2)} - \mathbf{x}^{(1)} \right\| \tag{16}$$

is the center-to-center distance between the two balls. The beam normal and shear stiffnesses are

$$k_n = \frac{E}{L}, \quad k_s = \frac{G}{L} = \frac{E}{2(1+\nu)L} \tag{17}$$

where E is the Young's modulus, G is the shear modulus, and ν is the Poisson's ratio.

The force-displacement law for the beam force and moment consists of the following steps.

1. Update F_n:

$$F_n := F_n + k_n A \Delta\delta_n \tag{18}$$

   where A is the cross-sectional area, and $\Delta\delta_n$ is the relative normal-displacement increment of Eq. (7).

2. Update $\mathbf{F_s}$:

$$\mathbf{F_s} := \mathbf{F_s} - k_s A \Delta\boldsymbol{\delta}_\mathbf{s} \quad \left[F_{ss} := F_{ss} - k_s A \Delta\delta_{ss},\ F_{st} := F_{st} - k_s A \Delta\delta_{st} \right] \tag{19}$$

   where A is the cross-sectional area, and $\Delta\boldsymbol{\delta}_\mathbf{s}$ is the relative shear-displacement increment of Eq. (7).

3. Update M_t:

$$M_t := M_t - k_s J \Delta\theta_t \tag{20}$$

   where J is the cross-sectional polar moment of inertia, and $\Delta\theta_t$ is the relative twist-rotation increment of Eq. (7).

4. Update $\mathbf{M_b}$:

$$M_{bs} := M_{bs} + \Delta M_{bs}, \quad M_{bt} := M_{bt} + \Delta M_{bt}$$
$$\begin{Bmatrix} \Delta M_{bs} \\ \Delta M_{bt} \end{Bmatrix} = \left[\mathbf{t_{gl}}\right]^T \begin{bmatrix} -k_n I_y & 0 \\ 0 & -k_n I_z \end{bmatrix} \left[\mathbf{t_{gl}}\right] \begin{Bmatrix} \Delta\theta_{bs} \\ \Delta\theta_{bt} \end{Bmatrix} \tag{21}$$

where I_y and I_z are the cross-sectional moments of inertia about the *y*- and *z*-axes, respectively, $\left[\mathbf{t_{gl}}\right]$ is the transformation matrix of Eq. (11), and $\Delta\boldsymbol{\theta}_\mathbf{b}$ is the relative bend-rotation increment of Eq. (7).

1.7 Properties

The property information is separated into parameters and state variables such that the parameters define the model, while the state variables describe its current state. The properties table provides a concise property reference that combines the parameters and state variables. The property information for the beam contact model is given in Tables 2–4.

Table 2 Beam Model Parameters

Parameter	Keyword	Description
	beam	model name
E	**bm_E**	Young's modulus
ν	**bm_nu**	Poisson's ratio
C_p	**bm_bhGiven**	cross-sectional properties code $\begin{cases} 0, & \text{specify directly} \\ 1, & \text{specify } b \text{ and } h \end{cases}$
A	**bm_A**	cross-sectional area $(C_p = 0)$
J	**bm_J**	polar moment of inertia $(C_p = 0)$
I_y	**bm_Iy**	moment of inertia about *y*-axis $(C_p = 0)$
I_z	**bm_Iz**	moment of inertia about *z*-axis $(C_p = 0)$
b	**bm_b**	cross-sectional width aligned with *y*-axis $(C_p = 1)$
h	**bm_h**	cross-sectional height aligned with *z*-axis $(C_p = 1)$
$\mathbf{Y}$	**bm_Ydir**	vector whose projection onto contact plane defines the beam *y*-axis

Table 3 Beam Model State Variables

Variable	Keyword	Description
L	**bm_L**	length
k_n	**bm_kn**	normal stiffness [stress/ disp.]
k_s	**bm_ks**	shear stiffness [stress/ disp.]
$\hat{\mathbf{y}}_b$	**bm_Yaxis**	beam coordinate system (*y*-axis)
$\mathbf{F}$	**bm_force**	beam force
$\mathbf{M}$	**bm_moment**	beam moment
E_k	**bm_Estr**	beam strain energy

Table 4 Beam Model Properties

Keyword	Symbol	Range	Default	Type	Modifiable
bm_E	E	$[0.0,\infty)$	0.0	FLT	yes
bm_nu	ν	$(-1.0,0.5]$	0.0	FLT	yes
bm_bhGiven	C_p	$\{0,1\}$	0	INT	yes
bm_A	A	$[0.0,\infty)$	0.0	FLT	yes
bm_J	J	$[0.0,\infty)$	0.0	FLT	yes
bm_Iy	I_y	$[0.0,\infty)$	0.0	FLT	yes
bm_Iz	I_z	$[0.0,\infty)$	0.0	FLT	yes
bm_b	b	$[0.0,\infty)$	0.0	FLT	yes
bm_h	h	$[0.0,\infty)$	0.0	FLT	yes
bm_Ydir	$\mathbf{Y}$	$[\mathbb{R},\mathbb{R},\mathbb{R}]$	$(0,1,0)$	VEC3	yes
bm_L	L	$[0.0,\infty)$	NA	FLT	no
bm_kn	k_n	$[0.0,\infty)$	0.0	FLT	no
bm_ks	k_s	$[0.0,\infty)$	0.0	FLT	no

bm_Yaxis	$\hat{\mathbf{y}}_b$	$[\mathbb{R},\mathbb{R},\mathbb{R}]$	NA	VEC3	no
bm_force	$\mathbf{F}$	$[\mathbb{R},\mathbb{R},\mathbb{R}]$	$\mathbf{0}$	VEC3	no
bm_moment	$\mathbf{M}$	$[\mathbb{R},\mathbb{R},\mathbb{R}]$	$\mathbf{0}$	VEC3	no
bm_Estr	E_k	$[0.0,\infty)$	0	FLT	no

1.8 Energies

The beam contact model stores the strain energy as a property. The beam strain energy is updated:

$$\mathrm{E}_k = \frac{1}{2}\left(\frac{F_n^{\ 2}}{k_n A} + \frac{\left\| F_s \right\|^2}{k_s A} + \frac{M_t^{\ 2}}{k_s J} + \frac{M_{by}^{\ \ 2}}{k_n I_y} + \frac{M_{bz}^{\ \ 2}}{k_n I_z} \right). \tag{22}$$

1.9 Methods

The beam contact model has no methods.

1.10 Time Step Estimation Scheme

The procedure to compute a stable time step (see Section 1.6 in the *Theory & Background* volume of Itasca [2008]) requires that each contact model return the contact translational and rotational stiffnesses. For the beam model, the translational stiffnesses are

$$\tilde{k}_n = k_n A, \quad \tilde{k}_s = k_s A \tag{23}$$

and the rotational stiffnesses are

$$\tilde{k}_t = k_s J, \quad \tilde{k}_b = k_n \max\left(I_y, I_z \right). \tag{24}$$

2.0 TEST PROBLEM

A test problem of a tip-loaded cantilever beam is provided for the beam contact model.

2.1 Tip-Loaded Cantilever Beam

The cantilever beam test problem is in the **`BeamContactModel\Test-Beam(rectXC)`** example-project directory. The problem is shown in Figures 6 and 7. The beam ends are denoted by 1 and 2. The beam is fully fixed at end 1, and different deformations are imposed at end 2 to provide three loading cases corresponding with axial, twisting, and flexural deformations. The closed-form expressions (McGuire et al., 1979) shown in the figures give the force and moment at the beam ends for each loading case.

The beam is comprised of polypropylene material with a density (ρ), Young's modulus (E), and Poisson's ratio (ν) of 946 kg/m^3, 1.2 GPa, and 0.42, respectively. The beam has a length (L) of 44 mm, and a rectangular cross section with a width (b) and height (h) of 4 and 1 mm, respectively. The cross-sectional properties of the rectangular cross section are given by Eq. (9). The closed-form expressions give the forces and moments at the beam ends, which are compared with the *PFC3D* model response below.

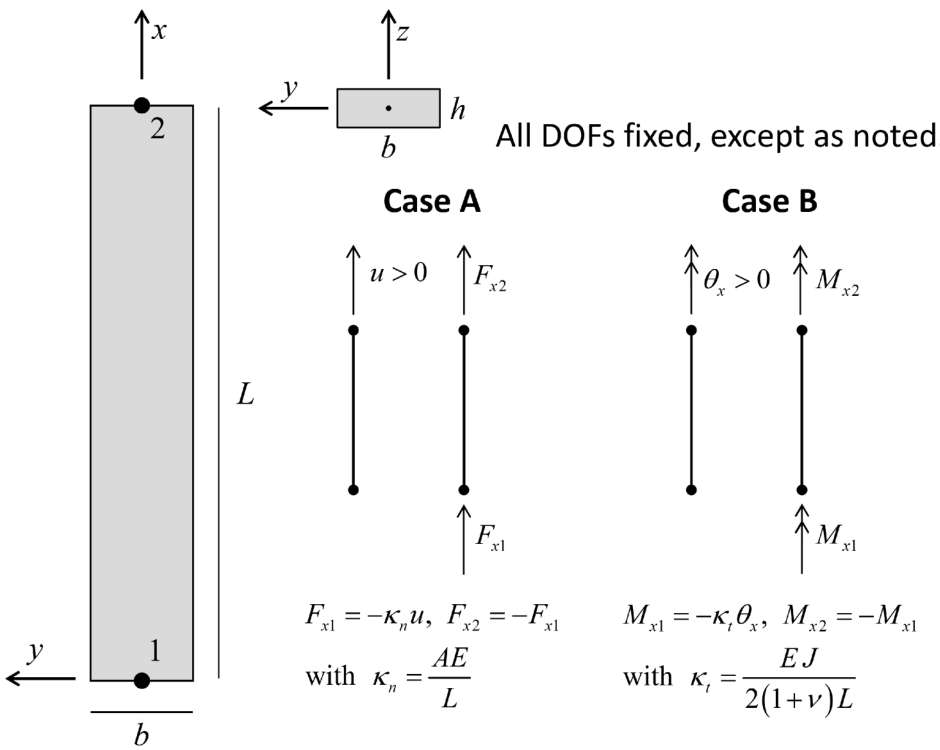

Figure 6 Cantilever beam test problem with loading cases for axial and twisting deformation.

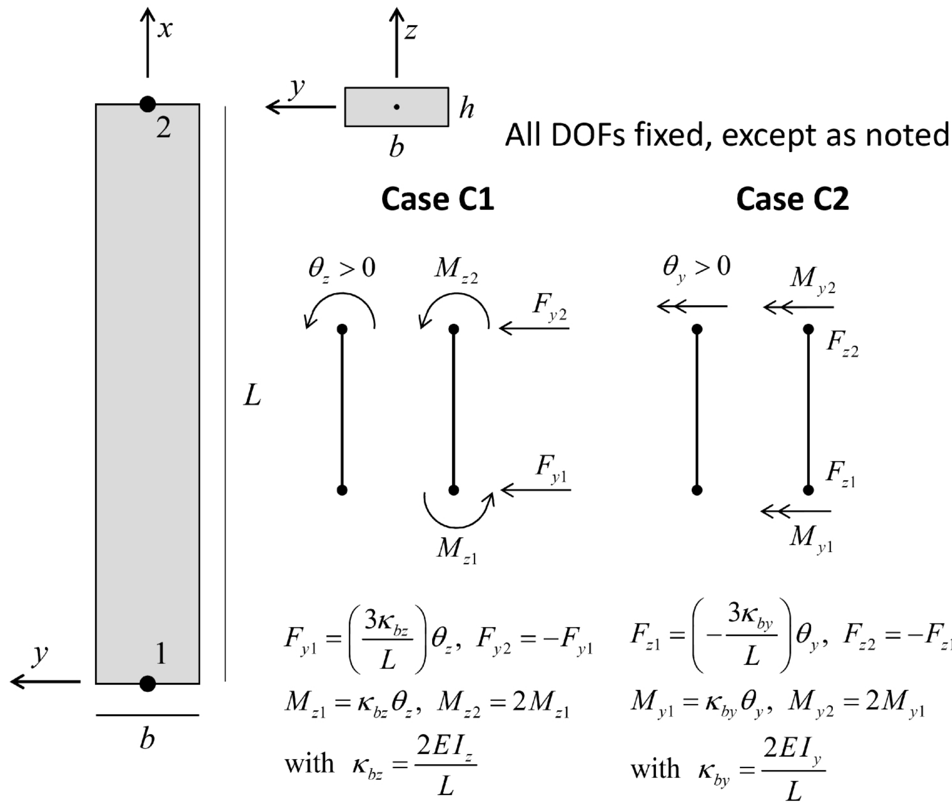

Figure 7 Cantilever beam test problem with loading cases for flexural deformation about the two principal axes of the beam cross section.

The beam is modeled as a collection of 12 balls joined by beam contacts (see Figure 8). All balls have a 4-mm diameter such that the end balls lie at the beam ends, and the 10 interior balls are placed next to one another with no overlap. The elastic constants, width and height of the beam cross section, and alignment of the *y*-axis are specified by assigning all beam contacts the properties:

$$
\begin{aligned}
&E = 1.2 \text{ GPa}, \quad \nu = 0.42 \\
&C_p = 1, \quad b = 4 \text{ mm}, \quad h = 1 \text{ mm}, \quad \mathbf{Y} = (0,1,0).
\end{aligned}
\tag{25}
$$

For this model, the beam y-axis corresponds with the global y-axis. The motion of the end balls is specified, and the interior balls are free to move in response to this imposed deformation. The end-1 ball is fully fixed, and the motion of the end-2 ball differs for each loading case.[7] After the end motions have been imposed, the model is cycled until static equilibrium is reached. The externally applied force and moment acting on the end balls are compared with the values from the closed-form solution for each loading case. The relative error for all cases is less than 3%. The flexural deformation of loading case C1 is evident in Figure 9, which shows the undeformed and deformed beams.

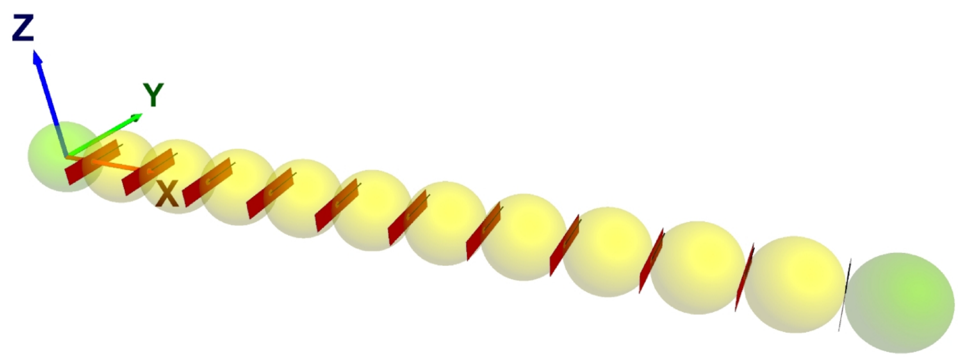

Figure 8 PFC3D model of cantilever-beam test problem showing the balls, beam interfaces, and global axes.

$$\text{Case A:} \quad u = 1 \text{ mm}$$

$$F_{x1} = \begin{cases} -1.0909 \times 10^{2} \text{ N}, & \text{analytical} \\ -1.0909 \times 10^{2} \text{ N}, & \text{numerical} \\ 0, & \text{relative error} \end{cases} \tag{26}$$

$$\text{Case B:} \quad \theta_x = 0.2 \text{ rad}$$

$$M_{x1} = \begin{cases} -1.0883 \times 10^{-2} \text{ Nm}, & \text{analytical} \\ -1.0883 \times 10^{-2} \text{ Nm}, & \text{numerical} \\ 0, & \text{relative error} \end{cases} \tag{27}$$

[7] For the flexural loading cases, the end-2 ball is freed in the x-direction to allow the beam to contract in the x-direction. If this is not done, then an axial force develops in the beam, and the PFC3D model exhibits stress stiffening such that the closed-form solution based on simple beam theory is no longer valid.

Case C1: $\theta_z = 0.2$ rad

$$F_{y1} = \begin{cases} 3.9669 \text{ N}, & \text{analytical} \\ 3.9113 \text{ N}, & \text{numerical} \\ 1.4\%, & \text{relative error} \end{cases} \tag{28}$$

$$M_{z1} = \begin{cases} 5.8182 \times 10^{-2} \text{ Nm}, & \text{analytical} \\ 5.6846 \times 10^{-2} \text{ Nm}, & \text{numerical} \\ 2.3\%, & \text{relative error} \end{cases}, \quad M_{z2} = \begin{cases} 1.1636 \times 10^{-1} \text{ Nm}, & \text{analytical} \\ 1.1481 \times 10^{-1} \text{ Nm}, & \text{numerical} \\ 1.3\%, & \text{relative error} \end{cases}$$

Case C2: $\theta_y = 0.2$ rad

$$F_{z1} = \begin{cases} -2.4793 \times 10^{-1} \text{ N}, & \text{analytical} \\ -2.4988 \times 10^{-1} \text{ N}, & \text{numerical} \\ 0.8\%, & \text{relative error} \end{cases}$$

$$M_{y1} = \begin{cases} 3.6364 \times 10^{-3} \text{ Nm}, & \text{analytical} \\ 3.6718 \times 10^{-3} \text{ Nm}, & \text{numerical} \\ 1.0\%, & \text{relative error} \end{cases}, \quad M_{y2} = \begin{cases} 7.2727 \times 10^{-3} \text{ Nm}, & \text{analytical} \\ 7.2942 \times 10^{-3} \text{ Nm}, & \text{numerical} \\ 0.3\%, & \text{relative error} \end{cases} \tag{29}$$

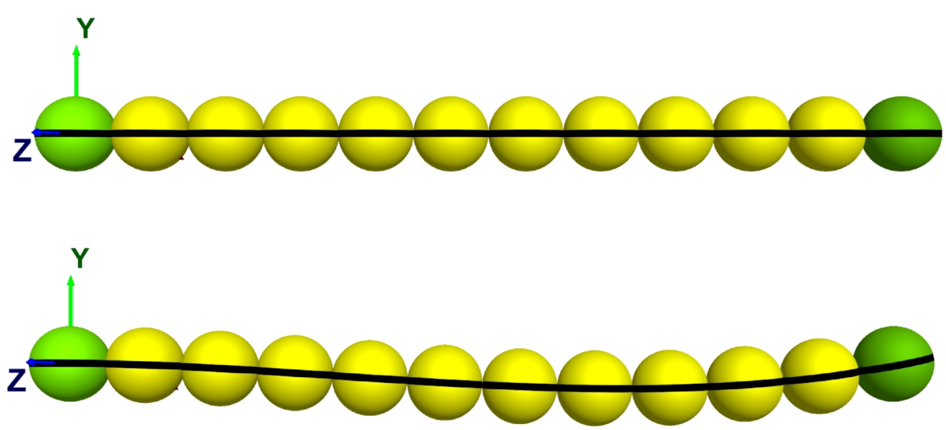

Figure 9 ***PFC3D model of cantilever-beam test problem at the start (top) and end (bottom) of loading case C1.***

3.0 REFERENCES

Itasca Consulting Group, Inc. (2019) *PFC — Particle Flow Code in 2 and 3 Dimensions*, Version 6.0, Documentation Set of version 6.00.12 [April 7, 2019]. Minneapolis: Itasca.

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